Chapter Eleven

Type I sums

Here, we must bound sums of the basic type

$$\sum_{m\leq D}\mu(m)\sum_{n}e(\alpha mn)\eta\left(\frac{mn}{x}\right) \tag{11.1}$$

and variants thereof.

There are three main improvements in comparison to standard treatments:

1. The terms with m divisible by q (and m not too large) will be taken out and treated separately by analytic means later. This all but eliminates what would otherwise be the main term.
2. The other terms get handled by improved estimates on trigonometric sums (§11.2). What is particularly important is to know which estimates to use for which m. For large m, the improvements have a substantial total effect – more than a constant factor is gained.
3. The "error" term $\delta/x = \alpha - a/q$ is used to our advantage. This happens both through the Poisson summation formula and through the use of two alternative approximations to the same number α.

We also use a continuous weight η, as is commonplace in our days in analytic number theory (though perhaps not quite commonplace enough). The improvements due to smoothing in type I are both relatively minor and essentially independent of the improvements due to (1) and (3), except for the fact that we do use continuity assumptions to ensure decay in the Poisson summation formula. The use of a continuous weight combines nicely with (2), but the ideas given here would give qualitative improvements in the treatment of trigonometric sums even in the absence of smoothing.

Since the main term we take out is exactly that, and not simply an error term to be bounded, we will be able to obtain cancellation in it later, and will also make this cancellation better by further smoothing. To wit: we will later need estimates on sums with a continuous truncation on m, rather than a brutal truncation $m \leq D$ as in (11.1). As usual, it is possible to pass from estimates for the brutal truncation to estimates for a continuous truncation; moreover, since the continuous function used for the truncation will be monotonic, we will not incur a loss in the passage. As will become clear at a later point, the main term will itself benefit from the passage to a smooth truncation.

* * *

An aside for specialists. For the purposes of this remark, a *specialist* is someone who has just pencilled in a note on the margin next to item 3 above, stating more or less the following: "we have known since long how to take advantage of large δ – just increase Q". What would be meant is the following: if $\alpha = a/q + \delta/x$, and we require that our approximation a/q to α satisfy $\alpha = a/q + O^*(1/qQ)$, then, for Q large enough, the approximation a/q will not be valid, and will be replaced by a different approximation a'/q', $q' \leq Q$, furnished by Dirichlet's approximation theorem. If q was too small, then the new q' will generally be much larger than q, and that is generally better, unless q' is too large.

In the first version of the proof, there was an important difficulty in applying such an approach in our case. All factors of $\log x$ had been removed from terms proportional to x/q – indeed, this is the main point of this chapter, and part of what makes the entire proof possible – but the same was not quite true of terms proportional to q. In the current version, as we will see, for large q, the terms proportional to q are free of factors of $\log x$ as well.

This means that we could prove our basic type I bound (Lemma 11.10) without showing how to take advantage of δ, and then proceed as we have just said to derive a bound that does take advantage of δ. We will discuss this possibility. As we shall see, however, our procedure, which takes advantage of δ directly, gives bounds that are at least about as good.

We will apply an idea related to the above *within* the proof of Lemma 11.10. We already set out how we switch approximations a/q, a'/q' in Lemma 2.2. We will split a crucial type I sum so as to use possibly distinct approximations a/q, a'/q' – whichever one is more useful – in different parts of the sum. We start to explain this in the following overview.

In the end, the precise road taken is in part a matter of taste. Keeping track of δ does turn out to be crucial in the bounds we prove *using* our basic type I bound, as well as in our type II bounds. The alternative – to allow very large q throughout – would be very unwieldy to say the least; some of the techniques we shall use for saving crucial factors of $\log$ do not seem to work in that case.

11.1 OVERVIEW

There are two type I sums, namely,

$$\sum_{\substack{m\leq U\\ m \text{ odd}}} \mu(m) \sum_{\substack{n\\ n \text{ odd}}} (\log n) e(\alpha m n) \eta\left(\frac{mn}{x}\right) \tag{11.2}$$

and

$$\sum_{\substack{v\leq V\\ v \text{ odd}}} \Lambda(v) \sum_{\substack{u\leq U\\ u \text{ odd}}} \mu(u) \sum_{n} e(\alpha v u n) \eta\left(\frac{vun}{x}\right). \tag{11.3}$$

In either case, $\alpha = a/q + \delta/x$, where q is larger than a constant r and $|\delta/x| \leq 1/qQ_0$ for some $Q_0 > \max(q, \sqrt{x})$. In this brief overview, we will estimate the slightly simpler sum (11.1), for the sake of exposition. There, D can be U or UV or something else less than x.

Why can we consider this simpler sum without omitting anything essential? It is clear that (11.2) is of the same kind as (11.1). The inner double sum in (11.3) is just (11.1) with αv instead of α and x/v instead of x. Hence, we can estimate (11.3) by means of (11.1) when q is small, that is, in the more delicate case. If q is not small, then the approximation $\alpha v \sim av/q$ may not be accurate enough. In that case, we collapse the two outer sums in (11.3) into a sum $\sum_n (\Lambda_{\leq V} * \mu_{\leq U})(n)$, and treat all of (11.3) much as we treat (11.1); since q is not small, we can afford to bound $(\Lambda_{\leq V} * \mu_{\leq U})(n)$ trivially (by $\log n$) in the less sensitive terms.

Let us see how to estimate (11.1), then, where $\alpha = a/q + \delta/x$ and $|\delta/x| \leq 1/q^2$. The basic procedure goes back to Vinogradov. He started as follows [Vin54, Ch. I, Lemma 5]:

$$\left|\sum_{n=1}^{N} e(\alpha n)\right| = \left|\frac{e(\alpha(N+1)) - e(\alpha)}{e(\alpha) - 1}\right| \leq \frac{1}{|\sin \pi\alpha|} \tag{11.4}$$

and then used the fact that $|\sin \pi\alpha|$ is bounded from below by twice the distance $d(\alpha, \mathbb{Z})$ of α to the nearest integer. Of course, he also had at his disposal a trivial bound – namely, the left side of (11.4) is at most N. Thus,

$$\left|\sum_{n=1}^{N} e(\alpha n)\right| \leq \min\left(N, \frac{1}{2d(\alpha, \mathbb{Z})}\right).$$

The next step is to split the outer sum in

$$\sum_{m \leq D} \mu(m) \sum_{n \leq x/m} e(\alpha m n)$$

into sums of length q. (Note this is Vinogradov's sum; we will have a smoothing factor $\eta(mn/x)$ instead of the condition $n \leq x/m$.) When m runs on an interval of length q, the angle am/q runs through all fractions of the form b/q; due to the error δ/x, αm could be close to 0 for two values of n, but otherwise $d(\alpha m, \mathbb{Z})$ takes values bounded below by $1/q$ (twice), $2/q$ (twice), $3/q$ (twice), etc. Thus

$$\begin{aligned}&\left|\sum_{y < m \leq y+q} \mu(m) \sum_{n \leq x/m} e(\alpha m n)\right| \leq \sum_{y < m \leq y+q} \left|\sum_{n \leq x/m} e(\alpha m n)\right| \\ &\leq 2\min\left(\frac{x}{y}, q\right) + 2\sum_{r=1}^{\lceil \frac{q-2}{2} \rceil} \frac{1/2}{r/q} \leq 2\min\left(\frac{x}{y}, q\right) + \sum_{r=1}^{\frac{q-1}{2}} \frac{q}{r} \leq 2\min\left(\frac{x}{y}, q\right) + q\log q\end{aligned} \tag{11.5}$$

for any $y \geq 0$, where we use (3.29). Hence, the expression in (11.1) is at most

$$\begin{aligned}&\sum_{r=1}^{\lfloor D/q \rfloor} \frac{2x}{rq} + \left(\frac{D}{q} + 1\right) \cdot q\log q + 2q \\ &\leq \frac{x}{q}\log\frac{D}{q} + D\log q + q(1 + 2\log q).\end{aligned} \tag{11.6}$$

We need to improve this qualitatively: we need to get, roughly, x/q *divided* by a log, not multiplied by a log, and, likewise, the terms $D\log q$ and $q(1+2\log q)$ have a factor of $\log q$ too many. We also need to optimize constants carefully, using the inputs given by the smoothing η.

The estimates from §3.1.2 and §3.2 on sums with smoothing give us that

$$\left|\sum_{n\leq N} e(\alpha m n)\eta\left(\frac{mn}{x}\right)\right| \leq \begin{cases} |\eta|_1 \frac{x}{m} + \frac{|\eta''|_1}{16}, \\ \frac{|\eta'|_1}{2|\sin(\pi m\alpha)|}, \\ \frac{|\widehat{\eta''}|_\infty}{4(\sin \pi m\alpha)^2}\,\frac{m}{x}. \end{cases} \tag{11.7}$$

We choose carefully which of the bounds on the right of (11.7) to use for which m; if we do so correctly, the term $q\log q$ in (11.5) becomes $O(q)$, and the term $D\log q$ in (11.6) becomes $O(D)$. The basic estimate here is Lemma 11.3, which, combined with (11.7), gives us – among other bounds – that

$$\sum_{y<m\leq y+q}\left|\sum_{n\leq x/m} e(\alpha m n)\right| \leq 3c_1|\eta|_1\frac{x}{y} + \frac{2q}{\pi}\sqrt{c_1|\eta|_1|\widehat{\eta''}|_\infty}, \tag{11.8}$$

where $c_1 = |\eta''|_1 y/8x$.

We also give versions of (11.8) in which the term corresponding to m divisible by q has been taken out. For m small – meaning: for $m \leq M$, where $M = \min(D, x/2q|\delta|)$ – the term proportional to x/y disappears in consequence. If m is large, then it does not make sense to take out the terms with m divisible by q, since those may not be the terms for which $m\alpha$ is close to 0; we will later see what to do.

At any rate, we are left with the sum of the terms with m small and divisible by q:

$$\left|\sum_{\substack{m\leq M\\ q|m}} \mu(m) \sum_{n\leq x/m} e(\alpha m n)\eta(mn/x)\right|.$$

We can estimate the inner sum by the Poisson summation formula, and then sum over m; writing $m = aq$, we get a main term

$$\frac{x\mu(q)}{q}\cdot\widehat{\eta}(-\delta)\cdot\sum_{\substack{a\leq M/q\\ (a,q)=1}}\frac{\mu(a)}{a}. \tag{11.9}$$

This is what we do in §11.2, in several variants. We will later estimate sums such as that in (11.9) using the bounds by Ramaré et al. in (5.46)–(5.74).

What shall we do for $m > Q/2$? The bound we shall obtain in Lemma 11.4 for sums over ranges of the form $y < n \leq y + 2q$ contains a term proportional to $A \sim |\eta|_1 x/y$. This looks unpleasant – it adds a multiple of $(x\log x)/q$, which is too much by a factor of $\log^2 x$ or so – and, at first sight, unavoidable: the values of m for which αm is close to 0 no longer correspond to the congruence class $m \equiv 0 \bmod q$, and thus cannot be taken out with ease.

The solution is to consider different approximations to α. What does this mean? If α were exactly, or almost exactly, a/q, then there would be no other very good approximations in a reasonable range. However, note that we can *define* $Q = x/|\delta q|$ for $\alpha = a/q + \delta/x$. If δ is very small, Q will be larger than $2D$, and there will be no terms with $Q/2 < m \leq D$ to worry about.

What happens if δ is not very small? We use Lemma 2.2, and obtain an approximation a'/q' to α with $Q/2 < q' \leq Q$. Then, for $m > Q/2$, we apply bounds such as (11.8) with a'/q' instead of a/q. The contribution of the term proportional to x/y is now insignificant: for the first sum over a range $y < m \leq y + q'$, $y \geq Q/2$, it is at most $x/(Q/2)$, and the sum over all following ranges is at most a constant times $(x \log x)/q'$.

Proceeding in this way, we obtain a total bound for (11.1) whose main terms are proportional to

$$\frac{1}{\phi(q)} \frac{x}{\log \frac{x}{q}} \min\left(1, \frac{1}{\delta^2}\right), \qquad q \log q, \qquad q \log^+ \frac{D}{q}, \qquad D \tag{11.10}$$

with good, explicit constants. The first term comes from the estimates we will later use on sums such as (11.9).

We will have to give several variants on the basic procedure we have just discussed, corresponding to the variants of (11.1) we will have to bound.

11.2 TRIGONOMETRIC SUMS

The following lemmas on trigonometric sums are a replacement for Vinogradov's basic bound (11.5) and similar results in the literature. The precise reference here is [Vin54, Ch. I, Lemma 8a, (I)]; the best previous result in this direction is in the work of Daboussi and Rivat [DR01, Lemma 1]. Just as in [DR01], we will actually work with sums of trigonometric functions, rather than bound them by harmonic sums $\sum_{n \leq q/2} 1/n$.

The main idea is to switch between different types of approximation within the sum, rather than just choosing between bounding all terms either trivially (by A) or non-trivially (by $C/|\sin(\pi\alpha n)|^2$). There will also be improvements in our applications stemming from the fact that Lemmas 11.3 and Lemma 11.5 take quadratic ($|\sin(\pi\alpha n)|^2$) rather than only linear ($|\sin(\pi\alpha n)|$) inputs. These improved inputs come from the use of smoothing elsewhere. Another important feature is that we will give bounds (Lemmas 11.1, 11.2 and 11.5) where the term with n divisible by q has been removed; this allows us to treat this typically large term later, when we will find cancellation in the sum of all such terms. Lastly, we will also obtain gains by restricting to odd n.

The first bound we give looks extremely similar to [Vin54, Ch. I, Lemma 8a, (I)]. The main difference is that we set aside the term that makes the largest contribution, namely, the term with n divisible by q. We prove this lemma in part for expository purposes.

Lemma 11.1. *Let $\alpha = a/q + \beta/qQ$, $(a,q) = 1$, $|\beta| \leq 1$, $1 \leq q \leq Q$. Let $y \geq 0$, $0 < m < q$. If $y + m \leq Q/2$, then*

$$\sum_{\substack{y<n\leq y+m \\ q\nmid n}} \frac{1}{|\sin(\pi\alpha n)|} \leq q \log em. \tag{11.11}$$

Proof. The main point is that, as n ranges within $y_1 < n \leq y_2$, omitting the value divisible by q (if any), $an \bmod q$ visits each element of

$$\pm 1, \pm 2, \ldots, \pm\lfloor q/2 \rfloor \mod q$$

at most once.

Clearly, αn equals $an/q + (n/Q)\beta/q$; since $y_2 \leq Q/3$, this means that $|\alpha n - an/q| \leq 1/3q$ for $n \leq y_2$; moreover, again for $n \leq y_2$, the sign of $\alpha n - an/q$ remains constant.

For $|\beta| \leq \pi/2$, we can bound, coarsely, $\sin\beta \geq (2/\pi)\beta$. We conclude that

$$\begin{aligned}\sum_{\substack{y<n\leq y+m \\ q\nmid n}} \frac{1}{|\sin(\pi\alpha n)|} &\leq \sum_{r=1}^{\lceil m/2\rceil} \frac{\pi/2}{\frac{\pi}{q}\left(r-\frac{1}{2}\right)} + \sum_{r=1}^{\lfloor m/2\rfloor} \frac{\pi/2}{\frac{\pi}{q}r} \\ &= q\sum_{n=1}^{m} \frac{1}{n} \leq q(\log m + 1).\end{aligned}$$

□

Sums involving sin, cos, etc., are called *trigonometric sums*, as one might expect. Daboussi and Rivat proved a version of Lemma 11.1 with a better constant in front. They did so by estimating the trigonometric sum on the left side of (11.11) directly, rather than bound it by a harmonic sum first. (Pólya's and Schur's versions ([Pol18], [Sch18]) of Pólya-Vinogradov were, in the limit, better than (3.79) by a constant factor, in part for exactly the same reason.) We will estimate trigonometric sums directly, too. Let us begin by giving an estimate for a family of sums considered by Daboussi and Rivat, again setting aside the term with n divisible by q.

Lemma 11.2. *Let $\alpha = a/q+\beta/qQ$, $(a,q) = 1$, $|\beta| \leq 1$, $1 \leq q \leq Q$. Let $y_2 > y_1 \geq 0$. If $y_2 - y_1 \leq q$ and $y_2 \leq Q/2$, then*

$$\sum_{\substack{y_1<n\leq y_2 \\ q\nmid n}} \frac{1}{|\sin(\pi\alpha n)|} \leq 2\frac{q}{\pi}\log\frac{7q}{3}. \tag{11.12}$$

The optimal constant is better than $7/3$ but worse than 2. We actually do not care much about the constant here; we will do a bit of work on it in order to illustrate a routine procedure we shall follow later.

Proof. By the same argument as in the proof of Lemma 11.1, the left side of (11.12) is at most

$$\sum_{r=1}^{\lfloor q/2 \rfloor} \frac{1}{\sin \frac{\pi}{q}\left(r-\frac{1}{2}\right)} + \sum_{r=1}^{\lfloor q/2-1/2 \rfloor} \frac{1}{\sin \frac{\pi}{q} r}. \tag{11.13}$$

Since $t \mapsto 1/\sin(t)$ is convex for $t \in (0, \pi/2]$, we see, much as in (3.1) that, for $q \geq 3$, the expression in (11.13) is at most

$$\begin{aligned} &\frac{1}{\sin \frac{\pi}{2q}} + \int_1^{q/2} \frac{dt}{\sin \frac{\pi}{q} t} + \frac{1}{\sin \frac{\pi}{q}} + \int_{3/2}^{q/2} \frac{dt}{\sin \frac{\pi}{q} t} \\ &= \frac{1}{\sin \frac{\pi}{2q}} + \frac{1}{\sin \frac{\pi}{q}} + \frac{q}{\pi}\left(\log \cot \frac{\pi}{2q} + \log \cot \frac{3\pi}{4q}\right), \end{aligned} \tag{11.14}$$

where we use the fact that $-\log \cot(x/2)$ is an antiderivative of $1/\sin x$.

We just have to verify that, for all $t \in (0, \pi/8]$,

$$\frac{t}{\sin t} + \frac{t}{\sin 2t} + \frac{1}{2}\left(\log \cot t + \log \cot \frac{3t}{2}\right) \leq \log \frac{C}{t} \tag{11.15}$$

for a constant C to be set soon. We will do this by comparing expansions around $t = 0$.[1] The cases $q = 2$, $q = 3$ can be dealt with separately:

$$\frac{\pi/4}{\sin \frac{\pi}{4}} \leq \log \frac{2.385}{\pi/4}, \quad \frac{\pi/6}{\sin \frac{\pi}{6}} + \frac{\pi/6}{\sin \frac{\pi}{3}} \leq \log \frac{2.732}{\pi/6}.$$

By [AS64, (4.3.68)] and [AS64, (4.3.70)], for $t \in (-\pi, \pi)$,

$$\begin{aligned} \frac{t}{\sin t} &= 1 + \sum_{k \geq 0} a_{2k+1} t^{2k+2} = 1 + \frac{t^2}{6} + \dots \\ t \cot t &= 1 - \sum_{k \geq 0} b_{2k+1} t^{2k+2} = 1 - \frac{t^2}{3} - \frac{t^4}{45} - \dots, \end{aligned} \tag{11.16}$$

where $a_{2k+1} \geq 0$, $b_{2k+1} \geq 0$ for $k \geq 0$. Clearly

$$\frac{t}{\sin t} = 1 + \frac{t^2}{6} + c_1(t) t^4, \tag{11.17}$$

where

$$c_1(t) = \frac{1}{t^4}\left(\frac{t}{\sin t} - \left(1 + \frac{t^2}{6}\right)\right), \tag{11.18}$$

[1] Usually, to prove an inequality such as (11.15), we would use a series expansion just to deal with a neighborhood of a point at which some of our functions are not well-defined – namely, $t = 0$, in this case – and supplement it by the bisection method. (Actually, the ability to carry out automatically such a combined procedure to prove inequalities should be added to the desiderata in section 4.1.) Here, a series expansion turns out to be enough to prove the statement in the desired range.

which is an increasing function on $[0,\pi)$ because $a_{2k+1} \geq 0$ for all $k \geq 0$. We check that $c_1(3\pi/8) = 0.02276\ldots$. Hence, for $t \in (0, \pi/8]$,

$$\frac{t}{\sin t} + \frac{t}{\sin 2t} \leq \frac{3}{2} + \frac{t^2}{2} + 9c_1(\pi/2)t^4,$$

$$\frac{1}{2}\left(\log t \cot t + \log t \cot \frac{3t}{2}\right) < -\frac{\log 3/2}{2} - \frac{1}{2}\left(\frac{t^2}{3} + \frac{(3t/2)^2}{3}\right) = -\frac{\log 3/2}{2} - \frac{13t^2}{24}.$$

Note that $t^2/24 < 9c_1(3\pi/8)t^4$ for $0 < t \leq \pi/8$. Hence (11.15) holds with $C = \sqrt{2e^3/3}$ for $t \in (0, \pi/8]$.

We conclude that (11.14) is at most

$$\frac{2q}{\pi} \log \frac{2Cq}{\pi}. \tag{11.19}$$

Note that $2C/\pi = 2.32957\ldots$.

□

Let us now see how we will proceed when we cannot take out the term with n divisible by q or when doing so would not help. There will be two differences from the traditional treatment. One is that, rather than take as an input a bound inversely proportional to $\sin \pi\alpha n$, we will (as in [Tao14, §5.2]) use a bound proportional to $\sin^2 \pi\alpha n$, coming from our use of smoothing elsewhere. The other difference is that we use the bound A not only in the worst case, but whenever it is preferable to the bound proportional to $\sin^2 \pi\alpha n$. This is surprisingly helpful.

Lemma 11.3. *Let $\alpha = a/q + \beta/q^2$, $q \geq 1$, $(a,q) = 1$, $|\beta| \leq 1$. Then, for any $A, C \geq 0$,*

$$\sum_{y<n\leq y+q} \min\left(A, \frac{C}{|\sin(\pi\alpha n)|^2}\right) \leq \min\left(2A + \frac{6q^2}{\pi^2}C, 3A + \frac{4q}{\pi}\sqrt{AC}\right). \tag{11.20}$$

Proof. We start by letting $m_0 = \lfloor y \rfloor + \lfloor (q+1)/2 \rfloor$, $j = n - m_0$, so that j ranges in the interval $(-q/2, q/2]$. We can write

$$\alpha m_0 \equiv c/q + \delta_2 \bmod 1,$$

where $c \in \mathbb{Z}/q\mathbb{Z}$ and $|\delta_2| \leq 1/2q$. Then, for $y < n \leq y + q$,

$$\begin{aligned}\alpha n = \alpha(j + m_0) &\equiv \alpha j + \frac{c}{q} + \delta_2 \\ &\equiv \frac{aj}{q} + \frac{\beta j}{q^2} + \frac{c}{q} + \delta_2 \equiv \frac{aj + c}{q} + \delta_1(j) + \delta_2 \bmod 1,\end{aligned}$$

where $|\delta_1(j)| \leq 1/2q$. Since the left side of (11.20) remains the same if α is replaced by $-\alpha$, we can assume that $\delta_2 \geq 0$ without loss of generality. The variable $r = aj + c \bmod q$ occupies each residue class $\bmod q$ exactly once.

One option is to bound the terms corresponding to $r = 0, -1$ by A each and all the other terms by $C/|\sin(\pi\alpha n)|^2$. (This can be seen as the simple case; it will take us a page and a half just because we have to estimate all sums and all terms here with great care – as in [DR01], only more so.)

The terms corresponding to $r = -k$ and $r = k - 1$ $(2 \leq k \leq q/2)$ contribute at most

$$\frac{C}{\sin^2 \frac{\pi}{q}(k - \frac{1}{2} - q\delta_2)} + \frac{C}{\sin^2 \frac{\pi}{q}(k - \frac{3}{2} + q\delta_2)}$$

because $t \mapsto \sin^2(\pi t/q)$ is decreasing on $[0, q/2]$, and because $|\delta_1(j)| \leq 1/2q$, $0 \leq \delta_2 \leq 1/2q$. Since $x \mapsto \frac{1}{(\sin x)^2}$ is convex on $(0, \pi)$, this is at most

$$\frac{C}{\sin^2 \frac{\pi}{q}\left(k - \frac{1}{2}\right)} + \frac{C}{\sin^2 \frac{\pi}{q}\left(k - \frac{3}{2}\right)}.$$

If q is odd, we must also include a term

$$\frac{C}{\sin^2 \frac{\pi}{q}\left(\left\lfloor\frac{q}{2}\right\rfloor - \frac{1}{2} + q\delta_2\right)} \leq \frac{C}{\sin^2 \frac{\pi}{q}\left(\left\lfloor\frac{q}{2}\right\rfloor - \frac{1}{2}\right)}$$

corresponding to $r = k$, $k = \lfloor q/2 \rfloor$.

Hence, in total, the terms with $r \neq 0, -1$ contribute at most

$$\frac{C}{\sin^2 \frac{\pi}{2q}} + 2\sum_{2 \leq r \leq \frac{q}{2}} \frac{C}{\sin^2 \frac{\pi}{q}(r - 1/2)} \leq \frac{C}{\sin^2 \frac{\pi}{2q}} + 2\int_1^{q/2} \frac{C}{\sin^2 \frac{\pi}{q}x} dx, \tag{11.21}$$

where we use again the convexity of $x \mapsto 1/(\sin x)^2$. (We can assume $q > 2$, as otherwise we have no terms other than $r = 0, 1$.) Now

$$\int_1^{q/2} \frac{1}{\sin^2 \frac{\pi}{q}x} dx = \frac{q}{\pi} \int_{\frac{\pi}{q}}^{\frac{\pi}{2}} \frac{1}{\sin^2 u} du = \frac{q}{\pi} \cot \frac{\pi}{q}. \tag{11.22}$$

Much as in (11.17), we bound

$$\left(\frac{t}{\sin t}\right)^2 = 1 + \frac{t^2}{3} + c_2(t)t^4, \tag{11.23}$$

where $c_2(t) = ((t/\sin t)^2 - (1 + t^2/3))/t^4$ is an increasing function. We check that $c_2(\pi/4) = 0.073806\ldots$. Using (11.16), we get

$$\begin{aligned}\frac{t^2}{\sin^2 t} + t \cot 2t &\leq \left(1 + \frac{t^2}{3} + c_2\left(\frac{\pi}{4}\right)t^4\right) + \left(\frac{1}{2} - \frac{2t^2}{3} - \frac{8t^4}{45}\right)\\ &= \frac{3}{2} - \frac{t^2}{3} + \left(c_2\left(\frac{\pi}{4}\right) - \frac{8}{45}\right)t^4 \leq \frac{3}{2} - \frac{t^2}{3} \leq \frac{3}{2}\end{aligned}$$

for $t \in [0, \pi/4]$. Therefore,

$$\frac{1}{\sin^2 \frac{\pi}{2q}} + \frac{2q}{\pi} \cot \frac{\pi}{q} \leq \frac{6q^2}{\pi^2}$$

for all $q \geq 2$, and so the first bound in (11.20) holds.

The following is an alternative approach; it yields the other estimate in (11.20). We bound the terms corresponding to $r = 0$, $r = -1$, $r = 1$ by A each. We let $r = \pm r'$ for r' ranging from 2 to $\lfloor q/2 \rfloor$. We obtain that the sum is at most

$$3A + \sum_{s=-1,1} \sum_{2 \leq r' \leq \lfloor q/2 \rfloor} \min \left(A, \frac{C}{\sin^2 \frac{\pi}{q} \left(r' - \frac{1}{2} + sq\delta_2\right)} \right). \tag{11.24}$$

We bound a term $\min(A, C/\sin((\pi/q)(r' - 1/2 \pm q\delta_2))^2)$ by A if and only if $C/\sin((\pi/q)(r' - 1 \pm q\delta_2))^2 \geq A$.

(In other words, we are choosing whichever of the two bounds A, $C/|\sin(\pi\alpha n)|^2$ is better. This is hardly anything deep, but, without this observation, we would be in serious difficulties. The first bound in (11.20) is much too large when q is large. The traditional procedure – in place since Vinogradov – is to use a bound of the type $B/|\sin(\pi\alpha n)|$, but that introduces a factor of $\log q$, as in Lemma 11.2, and we cannot afford that. Of course, one can use a bound of the type $B/|\sin(\pi\alpha n)|$ *and* the observation we are using here; this is what was done in the first version of the preprint [Helb]. That would result in replacing $\log q$ by $\log x/y$; what we do here will amount to replacing $\log q$ by a constant.)

Every r' with $C/\sin((\pi/q)(r' - 1 \pm q\delta_2))^2 \geq A$ satisfies

$$r' \leq 1 + \frac{q}{\pi} \arcsin \sqrt{\frac{C}{A}} \mp q\delta_2.$$

Hence, the number of such terms in either of the sums in (11.24) is

$$\leq \max \left(0, \left\lfloor \frac{q}{\pi} \arcsin \sqrt{\frac{C}{A}} \mp q\delta_2 \right\rfloor \right)$$

(note the condition $r' \geq 2$ in (11.24)), and thus, in total over the two sums, the number of all such terms is $\leq (2q/\pi) \arcsin(\sqrt{C/A})$. (Recall that $q\delta_2 \leq 1/2$.) Each one of the other terms gets bounded by the integral of $C/\sin^2(\pi\alpha/q)$ from $\alpha = r' - 1 \pm q\delta_2$ ($> (q/\pi) \arcsin(\sqrt{C/A})$) to $\alpha = r' \pm q\delta_2$, by convexity. Thus (11.24) is at most

$$\begin{aligned} &3A + \frac{2q}{\pi} A \arcsin \sqrt{\frac{C}{A}} + \int_{\frac{q}{\pi} \arcsin \sqrt{\frac{C}{A}}}^{q/2 - q\delta_2} \frac{C}{\sin^2 \frac{\pi\alpha}{q}} d\alpha + \int_{\frac{q}{\pi} \arcsin \sqrt{\frac{C}{A}}}^{q/2 + q\delta_2} \frac{C}{\sin^2 \frac{\pi\alpha}{q}} d\alpha \\ &= 3A + \frac{2q}{\pi} A \arcsin \sqrt{\frac{C}{A}} + 2 \int_{\frac{q}{\pi} \arcsin \sqrt{\frac{C}{A}}}^{q/2} \frac{C}{\sin^2 \frac{\pi\alpha}{q}} d\alpha \\ &\leq 3A + \frac{2q}{\pi} A \arcsin \sqrt{\frac{C}{A}} + \frac{2q}{\pi} \sqrt{AC} \sqrt{1 - C/A} \end{aligned} \tag{11.25}$$

We can easily show (taking derivatives) that $\arcsin x + x\sqrt{1-x^2} \leq 2x$ for $0 \leq x \leq 1$. Setting $x = \sqrt{C/A}$, we see that this implies that

$$3A + \frac{2q}{\pi} A \arcsin\sqrt{\frac{C}{A}} + \frac{2q}{\pi} A \sqrt{\frac{C}{A}}\sqrt{1-\frac{C}{A}} \leq 3A + \frac{4q}{\pi}\sqrt{AC}$$

for $C/A \leq 1$. If $C/A > 1$, then $3A + (4q/\pi)\sqrt{AC}$ is greater than Aq, which is an obvious upper bound for the left side of (11.20). □

Let us now prove an analogue of Lemma 11.3 for a sum over odd values of n. We could, in principle, proceed as in [Tao14, Cor. 3.5] to derive such an analogue from Lemma 11.3 itself. However, this would lead to a bound that is sometimes substantially worse (and sometimes better) than the one we will derive – and it would also force us to work with 2α instead of α here, and with 4α instead of 2α elsewhere. This seems to increase casework in other sections, and thus we decide against it.

Lemma 11.4. *Let $\alpha = a/q + \beta/q^2$, $q \geq 1$, $(a, q) = 1$, $|\beta| \leq 1$. Then, for any $A, C \geq 0$,*

$$\sum_{\substack{y<n\leq y+2q \\ n \text{ odd}}} \min\left(A, \frac{C}{\sin^2(\pi\alpha n)}\right) \leq 3A + \min\left(\frac{6q^2}{\pi^2}C, \frac{4q}{\pi}\sqrt{AC}\right). \quad (11.26)$$

Proof. We let $m_0 = y + q$, $j = n - m_0$, so that j ranges on $(-q, q]$. Just as in the proof of Lemma 11.3, we can write

$$\begin{aligned}\alpha n = \alpha(j + m_0) &\equiv \alpha j + \frac{c}{q} + \delta_2 \\ &\equiv \frac{aj + c}{q} + \frac{\beta j}{q^2} + \delta_2 \mod 1,\end{aligned}$$

where $c \in \mathbb{Z}/q\mathbb{Z}$, $|\delta_2| \leq 1/2q$. As before, we can assume that $\delta_2 \geq 0$ without loss of generality. If q is odd, we let $\delta_1(j) = \beta j/q^2$ for all $-q < j \leq q$; if q is even, we let $\delta_1(j) = \beta j/2q^2$ for all $0 < j \leq q$, while, for $-q < j \leq 0$, we let $\delta_1(j) = \beta j/2q^2 - 1/q$ if $\beta \geq 0$, and $\delta_1(j) = \beta j/2q^2 + 1/q$ if $\beta < 0$.

Thus, for q even or odd, when j runs over all odd numbers in $(-q, q]$, we see that

$$\alpha(j + m_0) \equiv \frac{r_j}{q} + \delta_1(j) + \delta_2 \mod 1,$$

where $|\delta_1(j)| \leq 1/q$ and $r = r_j$ runs over all elements of $\mathbb{Z}/q\mathbb{Z}$ (exactly one time each).

We bound the terms corresponding to $0, -1, 1$ by A. We can bound the other terms by $C/\sin^2 \pi\alpha n$. The terms corresponding to $r = -(k+1)$ and $r = (k+1)$ $(1 \leq r \leq q/2 - 1)$ contribute at most

$$\min\left(A, \frac{C}{\sin^2 \frac{\pi}{q}(k - q\delta_2)}\right) + \min\left(A, \frac{C}{\sin^2 \frac{\pi}{q}(k + q\delta_2)}\right). \quad (11.27)$$

By the convexity of $x \mapsto 1/\sin^2 x$, this is at most

$$\frac{C}{\sin^2 \frac{\pi}{q}\left(k-\frac{1}{2}\right)} + \frac{C}{\sin^2 \frac{\pi}{q}\left(k+\frac{1}{2}\right)}.$$

Hence, the terms with $r \neq -1, 0, 1$ contribute at most

$$\frac{C}{\sin^2 \frac{\pi}{2q}} + 2 \sum_{2 \leq r \leq \frac{q}{2}} \frac{C}{\sin^2 \frac{\pi}{q}(r-1/2)},$$

which, as we already showed in the proof of Lemma 11.3, is at most $(6/\pi^2)Cq^2$. Therefore,

$$\sum_{\substack{y<n\leq y+2q \\ n \text{ odd}}} \min\left(A, \frac{C}{\sin^2(\pi\alpha n)}\right) \leq 3A + \frac{6q^2}{\pi^2}C. \tag{11.28}$$

At the same time, (11.27) also gives us the expression (11.24) as a bound for the left side of (11.28). We proceed exactly as in the proof of Lemma 11.3, and conclude that that expression is at most $3A + (4q/\pi)\sqrt{AC}$. □

Lastly, let us give a bound where, as in Lemma 11.2, we take out the contribution of n divisible by q, but where, as in Lemma 11.4, we restrict to n odd and take an input inversely proportional to $\sin^2$.

Lemma 11.5. *Let $\alpha = a/q + \beta/qQ$, $(a, q) = 1$, $|\beta| \leq 1$, $1 \leq q \leq Q$. Let $s \in \mathbb{Z}^+$, $y_1, y_2 \in \mathbb{R}$ be such that $sq - q \leq y_1 \leq y_2 < sq + q$ and $y_2 \leq Q/3$. Then*

$$\sum_{\substack{y_1<n\leq y_2 \\ n\neq sq \\ n \text{ odd}}} \frac{1}{\sin^2 \pi\alpha n} \leq \frac{8}{\pi^2}q^2. \tag{11.29}$$

This is actually fairly coarse, in that the proof below shows that we could replace the constant $8/\pi^2$ by $7.6244/\pi^2 = 0.7725\ldots$. We will, however, find $8/\pi^2$ convenient to work with later. Incidentally, getting a constant that is not worse than $8/\pi^2$ is the reason for having the condition $y_2 \leq Q/3$ here, rather than $y_2 \leq Q/2$. That is one of many small choices with minor advantages and disadvantages; a denominator of 3 instead of 2 here leads to having $\sqrt{7}$ instead of $\sqrt{5}$ in (11.52).

Proof. For $n \leq y_2$, $\alpha n - an/q$ is of constant sign and has absolute value bounded by $1/3q$ (since $y_2 \leq Q/3$). We can obviously assume that $q \geq 2$, since otherwise the sum we are trying to bound is empty.

If q is even, then, as n ranges over the odd numbers in $y_1 < n \leq y_2$, the residue $an \bmod q$ ranges over

$$\pm 1, \pm 3, \pm 5, \ldots, \pm(2j+1), \ldots$$

modulo q, for $2j+1 \leq q/2$, visiting each value twice. (If j is such that $2j+1 = q/2$, we count only $2j+1$, say, and not $-(2j+1)$.) Since $\alpha n - an/q$ is of constant sign, it follows that the left side of (11.29) is at most twice

$$\sum_{\substack{1\leq r\leq q/2\\ r \text{ odd}}} \frac{1}{\sin^2 \frac{\pi}{q}(r-1/3)} + \sum_{\substack{1\leq r\leq \frac{q}{2}-1\\ r \text{ odd}}} \frac{1}{\sin^2 \frac{\pi}{q} r}. \tag{11.30}$$

If q is odd, then the residue $an \bmod q$ ranges over

$$\pm 1, \pm 2, \pm 3, \ldots, \pm (q-1)/2,$$

visiting each value once. The left side of (11.29) is at most

$$\sum_{1\leq r\leq (q-1)/2} \frac{1}{\sin^2 \frac{\pi}{q}(r-1/3)} + \sum_{1\leq r\leq (q-1)/2} \frac{1}{\sin^2 \frac{\pi}{q} r}. \tag{11.31}$$

This is clearly bounded above by twice the expression in (11.30).

We check that, for $q=2$, the bound (11.30) is equal to $1/\sin^2(\pi/3) = 4/3 < 16/\pi^2 = 4q^2/\pi^2$; for $q=4$, it is $= 1/\sin^2(\pi/6) + 1/\sin^2(\pi/4) = 6 < 64/\pi^2 = 4q^2/\pi^2$. For $q=3$ and $q=5$, it is easy to check that the bound (11.31) is less than $8q^2/\pi^2$ (considerably so). We can thus assume from now on that $q \geq 6$.

Since $x \mapsto 1/\sin^2 x$ is convex, (11.30) is bounded by

$$\frac{1}{\sin^2 \frac{2\pi}{3q}} + \frac{1}{2}\int_{5/3}^{q/2} \frac{1}{\sin^2 \frac{\pi}{q} x} dx + \frac{1}{\sin^2 \frac{\pi}{q}} + \frac{1}{2}\int_{2}^{q/2} \frac{1}{\sin^2 \frac{\pi}{q} x} dx. \tag{11.32}$$

This equals

$$\begin{aligned}&\frac{1}{\sin^2 \frac{2\pi}{3q}} + \frac{q}{2\pi}\cot\frac{5\pi}{3q} + \frac{1}{\sin^2\frac{\pi}{q}} + \frac{q}{2\pi}\cot\frac{2\pi}{q}\\ &= \frac{q^2}{\pi^2}\left(\frac{9}{4}\frac{(2\pi/3q)^2}{\sin^2\frac{2\pi}{3q}} + \frac{3}{10}\frac{5\pi}{3q}\cot\frac{5\pi}{3q} + \frac{(\pi/q)^2}{\sin^2\frac{\pi}{q}} + \frac{1}{4}\frac{2\pi}{q}\cot\frac{2\pi}{q}\right).\end{aligned}$$

By (11.16) and (11.23), this is at most q^2/π^2 times

$$\begin{aligned}&\frac{9}{4}\left(1+\frac{4t^2/9}{3} + c_2(2t/3)\frac{(2t)^4}{3^4}\right) + \frac{3}{10}\left(1 - \frac{(5t/3)^2}{3} - \frac{(5t/3)^4}{45}\right)\\ &+ 1 + \frac{t^2}{3} + c_2(t)t^4 + \frac{1}{4}\left(1 - \frac{(2t)^2}{3} - \frac{(2t)^4}{45}\right)\\ &= \frac{38}{10} + \frac{1}{18}t^2 + \left(\frac{4}{9}c_2\left(\frac{\pi}{9}\right) + c_2\left(\frac{\pi}{6}\right) - \frac{341}{2430}\right)t^4 \leq \frac{38}{10} + \frac{t^2}{18} - 0.040433t^4,\end{aligned}$$

where $t = \pi/q \leq \pi/6$ and $c_2(t)$ is as in (11.23). Now, a derivative test shows that $t^2/18 - 0.040433t^4$ is increasing for $t \leq \pi/6$, and so its maximum for $0 \leq t \leq \pi/6$ is

$(\pi/6)^2/18 - 0.040433 \cdot (\pi/6)^4 \leq 0.0122$. We conclude that

$$\sum_{\substack{1\leq r\leq q/2\\ r \text{ odd}}} \frac{1}{\sin^2 \frac{\pi}{q}(r-1/3)} + \sum_{\substack{1\leq r\leq \frac{q}{2}-1\\ r \text{ odd}}} \frac{1}{\sin^2 \frac{\pi}{q} r} \leq \left(\frac{38}{10} + 0.0122\right)\frac{q^2}{\pi^2} \leq \frac{4}{\pi^2} q^2.$$

□

11.3 THE CONTRIBUTION OF THE ZERO MODULUS

Let us treat the terms of sums of the basic type

$$\sum_m \mu(m) \sum_n e(\alpha m n)\eta\left(\frac{mn}{x}\right) \tag{11.33}$$

coming from m small and divisible by q. Here "small" is defined relative to the error δ in the Diophantine approximation $\alpha = a/q + \delta/x$. If the error δ is small, then the range of m we will consider is actually rather large. The point is that m must be small enough for $m\delta/x$ to be at most $1/2q$, say; obviously "small enough" here depends on the size of δ.

Instead of giving just a bound for the inner sum in (11.33), we estimate the inner sum as a main term plus an error term. We isolate the contribution of the main term in a way that makes it clear that we shall obtain cancellation in the contribution, thanks to the factor of $\mu(m)$.

Lemma 11.6. *Let $\alpha = a/q + \delta/x$, $(a,q) = 1$. Let $M \leq x/2|\delta|q$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise C^1 function such that $\eta, \eta', \eta'' \in L^1$ and $\eta(t) = 0$ for $t \leq 0$. Then*

$$\sum_{\substack{m\leq M\\ q|m}} \mu(m) \sum_n e(\alpha m n)\eta\left(\frac{mn}{x}\right) \tag{11.34}$$

is

$$\frac{x\mu(q)}{q}\widehat{\eta}(-\delta)\cdot m_q\left(\frac{M}{q}\right) + O^*\left(|\widehat{\eta''}|_\infty \cdot \left(\frac{1}{4} - \frac{1}{\pi^2}\right)\left(\frac{M^2}{2xq} + \frac{M}{2x}\right)\right).$$

Here, as usual, $m_q(y) = \sum_{m\leq y:(m,q)=1} \mu(m)/m$. We will be isolating similar sums in the following lemmas, following the notation in (5.45). We will later bound such sums using the estimates in §5.3.4.

Proof. Let $m \leq M$ be divisible by q. Then $e(\alpha m n)$ equals $e((\delta m/x)n)$. By Poisson summation,

$$\sum_{n\in\mathbb{Z}} e(\alpha m n)\eta(mn/x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n),$$

where $f(u) = e((\delta m/x)u)\eta((m/x)u)$. Now

$$\widehat{f}(n) = \int e(-un) f(u) du = \frac{x}{m}\int e\left(\left(\delta - \frac{xn}{m}\right)u\right)\eta(u)du = \frac{x}{m}\widehat{\eta}\left(\frac{nx}{m} - \delta\right).$$

By assumption, $m \leq M \leq Q/2 \leq x/2|\delta q|$, and so $|x/m| \geq 2|\delta q| \geq 2\delta$. Thus, by (2.16) (with $k = 2$),

$$\begin{aligned}\sum_n \widehat{f}(n) &= \frac{x}{m}\left(\widehat{\eta}(-\delta) + \sum_{n\neq 0}\widehat{\eta}\left(\frac{nx}{m} - \delta\right)\right)\\ &= \frac{x}{m}\left(\widehat{\eta}(-\delta) + O^*\left(\sum_{n\neq 0}\frac{1}{\left(2\pi\left(\frac{nx}{m}-\delta\right)\right)^2}\right)\cdot\left|\widehat{\eta''}\right|_\infty\right)\\ &= \frac{x}{m}\widehat{\eta}(-\delta) + \frac{m}{x}\frac{|\widehat{\eta''}|_\infty}{(2\pi)^2}O^*\left(\max_{|r|\leq\frac{1}{2}}\sum_{n\neq 0}\frac{1}{(n-r)^2}\right).\end{aligned} \tag{11.35}$$

(This is, of course, a very well-known procedure in many settings: apply Poisson summation, pick the term $\widehat{f}(0)$, and treat all other terms as error terms.)

Since $x \mapsto 1/x^2$ is convex on $\mathbb{R}^+$,

$$\max_{|r|\leq\frac{1}{2}}\sum_{n\neq 0}\frac{1}{(n-r)^2} = \sum_{n\neq 0}\frac{1}{\left(n-\frac{1}{2}\right)^2} = \pi^2 - 4. \tag{11.36}$$

Therefore, $\sum_{m\leq M: q|m}\mu(m)\sum_n e(\alpha mn)\eta(mn/x)$ equals

$$\begin{aligned}&\sum_{\substack{m\leq M\\ q|m}}\mu(m)\frac{x}{m}\widehat{\eta}(-\delta) + O^*\left(\sum_{\substack{m\leq M\\ q|m}}|\mu(m)|\frac{m}{x}\frac{|\widehat{\eta''}|_\infty}{(2\pi)^2}(\pi^2-4)\right)\\ &= \frac{x\mu(q)}{q}\cdot\widehat{\eta}(-\delta)\cdot\sum_{\substack{m\leq\frac{M}{q}\\ (m,q)=1}}\frac{\mu(m)}{m}\\ &+ O^*\left(\mu(q)^2|\widehat{\eta''}|_\infty\cdot\left(\frac{1}{4}-\frac{1}{\pi^2}\right)\left(\frac{M^2}{2xq}+\frac{M}{2x}\right)\right).\end{aligned}$$

□

Lemma 11.6 is there to illustrate its argument. We will actually restrict our variables to the odd integers. Let us see what that restriction does to the inner sum.

Lemma 11.7. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$. Let a, q, $(a, q) = 1$, and δ be such that $2\alpha = a/q + \delta/x$, where $x \geq q$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise C^1 function such that $\eta, \eta', \eta'' \in L^1$ and $\eta(t) = 0$ for $t \leq 0$. Let $m \leq x/2|\delta|q$, $q|m$.*

Then

$$\sum_{n \text{ odd}} e(\alpha mn)\eta\left(\frac{mn}{x}\right) = \kappa\frac{x}{2m}\widehat{\eta}\left(-\frac{\delta}{2}\right) + O^*\left(\frac{m}{x}\frac{|\widehat{\eta''}|_\infty}{2\pi^2}(\pi^2-4)\right), \tag{11.37}$$

where $\kappa = (-1)^{a'}$ and a' is such that $\alpha = a'/2q + \delta/2x$.

Proof. For $n = 2r + 1$,

$$\begin{aligned} e(\alpha mn) &= e(\alpha m(2r+1)) = e(2\alpha rm)e(\alpha m) \\ &= e\left(\frac{\delta}{x} rm\right) e\left(\left(\frac{a'}{2q} + \frac{\delta}{2x}\right) m\right) \\ &= e\left(\frac{\delta(2r+1)}{2x} m\right) e\left(\frac{a'}{2}\frac{m}{q}\right) = \kappa e\left(\frac{\delta(2r+1)}{2x} m\right), \end{aligned}$$

where $\kappa = e(a'/2) \in \{-1, 1\}$ is independent of m and n. Hence, by Poisson summation (as in (3.14)),

$$\begin{aligned} \sum_{n \text{ odd}} e(\alpha mn)\eta(mn/x) &= \kappa \sum_{n \text{ odd}} e((\delta m/2x)n)\eta(mn/x) \\ &= \frac{\kappa}{2}\left(\sum_n \widehat{f}(n) - \sum_n \widehat{f}(n + 1/2)\right), \end{aligned} \tag{11.38}$$

where $f(u) = e((\delta m/2x)u)\eta((m/x)u)$. Now

$$\widehat{f}(t) = \frac{x}{m}\widehat{\eta}\left(\frac{x}{m}t - \frac{\delta}{2}\right).$$

Since $m \leq M \leq x/2|\delta|q$, we know that $|x/m| \geq 2|\delta q| \geq 2\delta$. Thus, using the bounds in (2.16) and (11.36), we see that

$$\begin{aligned} &\frac{1}{2}\left(\sum_n \widehat{f}(n) - \sum_n \widehat{f}(n + 1/2)\right) \\ &= \frac{x}{m}\left(\frac{1}{2}\widehat{\eta}\left(-\frac{\delta}{2}\right) + \frac{1}{2}O^*\left(\sum_{n \neq 0}\left|\widehat{\eta}\left(\frac{x}{m}\frac{n}{2} - \frac{\delta}{2}\right)\right|\right)\right) \\ &= \frac{x}{m}\left(\frac{1}{2}\widehat{\eta}\left(-\frac{\delta}{2}\right) + \frac{1}{2}\cdot O^*\left(\sum_{n \neq 0}\frac{1}{\left(\pi\left(\frac{nx}{m} - \delta\right)\right)^2}\right)\cdot\left|\widehat{\eta''}\right|_\infty\right) \\ &= \frac{x}{2m}\widehat{\eta}\left(-\frac{\delta}{2}\right) + O^*\left(\frac{m}{x}\frac{|\widehat{\eta''}|_\infty}{2\pi^2}(\pi^2 - 4)\right). \end{aligned} \tag{11.39}$$

□

Here is a variant of Lemma 11.6 for later use.

Lemma 11.8. *Let* $2\alpha = a/q + \delta/x$, $(a, q) = 1$, $q \leq x$. *Let* $M \leq \min(x/2|\delta|q, x/e)$. *Let* $\eta : \mathbb{R} \to \mathbb{R}$ *be a continuous, piecewise* C^1 *function such that* $\eta, \eta', \eta'' \in L^1$ *and* $\eta(t) = 0$ *for* $t \leq 0$. *Let* $\eta_*(t) = \eta(t)\log t$ *for* $t > 0$, $\eta_*(0) = 0$ *for* $t \leq 0$. *Assume that* η_* *is continuous and that* $\eta_*, \eta_*', \eta_*'' \in L^1$.

Then

$$\sum_{\substack{m\le M\\ m \text{ odd}\\ q|m}} \mu(m) \sum_{n \text{ odd}} (\log n) e(\alpha m n) \eta\left(\frac{mn}{x}\right) \tag{11.40}$$

equals 0 *if* q *is even, and* κ *times*

$$\frac{x\mu(q)}{2q}\widehat{\eta}\left(-\frac{\delta}{2}\right)\cdot \check{m}_{2q}\left(\frac{M}{q}\right) + \frac{x\mu(q)}{2q}\left(\widehat{\eta_*}\left(-\frac{\delta}{2}\right) + \widehat{\eta}\left(-\frac{\delta}{2}\right)\log\frac{x}{M}\right) m_{2q}\left(\frac{M}{q}\right)$$
$$+\left(\frac{1}{8}-\frac{1}{2\pi^2}\right) O^*\left(|\widehat{\eta_*''}|_\infty\left(\frac{M^2}{qx}+3\right) + |\widehat{\eta''}|_\infty\left(\frac{M^2}{qx}\log\frac{\sqrt{e}x}{M}+\frac{4}{e}\right)\right)$$

if q *is odd. Here* $\kappa = (-1)^{a'}$, *where* a' *is such that* $\alpha = a'/2q + \delta/2x$.

Here $\check{m}_q$ is as in (5.45).

Proof. We apply Lemma 11.7, using $\eta_{(x/m)}$ instead of η, where

$$\eta_{(\rho)}(t) := (\log \rho)\eta(t) + \eta_*(t) = \begin{cases} \log(\rho t)\eta(t) & \text{if } t > 0,\\ 0 & \text{if } t \le 0.\end{cases} \tag{11.41}$$

We obtain that $\sum_{n \text{ odd}} (\log n) e(\alpha n)$ equals

$$\kappa \frac{x}{2m}\widehat{\eta_{(x/m)}}\left(-\frac{\delta}{2}\right) + O^*\left(\frac{m}{x}\frac{|\widehat{\eta''_{(x/m)}}|_\infty}{2\pi^2}(\pi^2-4)\right). \tag{11.42}$$

Now

$$\widehat{\eta_{(\rho)}} = (\log\rho)\widehat{\eta} + \widehat{\eta_*},$$
$$\widehat{\eta''_{(\rho)}} = (\log\rho)\widehat{\eta''} + \widehat{\eta''_*}.$$

Hence,

$$\sum_{\substack{m\le M\\ m \text{ odd}\\ q|m}} \mu(m)\frac{x}{2m}\widehat{\eta_{(x/m)}}\left(-\frac{\delta}{2}\right)$$

equals

$$\frac{x}{2}\widehat{\eta}\left(-\frac{\delta}{2}\right)\sum_{\substack{m\le M\\ m \text{ odd}\\ q|m}} \frac{\mu(m)}{m}\log\frac{x}{m} + \frac{x}{2}\widehat{\eta_*}\left(-\frac{\delta}{2}\right)\sum_{\substack{m\le M\\ m \text{ odd}\\ q|m}} \frac{\mu(m)}{m},$$

which is

$$\frac{x\mu(q)}{2q}\cdot\widehat{\eta}\left(-\frac{\delta}{2}\right)\sum_{\substack{m\le M/q\\ (m,2q)=1}} \frac{\mu(m)}{m}\log\frac{x}{mq}$$
$$+\frac{x\mu(q)}{2q}\cdot\widehat{\eta_*}\left(-\frac{\delta}{2}\right)\sum_{\substack{m\le M/q\\ (m,2q)=1}} \frac{\mu(m)}{m}.$$

for q odd, and vanishes for q even. We reassemble the sums using the fact that $\log(x/m) = \log(M/m) + \log(x/M)$.

It remains to estimate

$$\sum_{\substack{m\le M\\ m \text{ odd}\\ q|m}} \frac{m}{x} \frac{|\widehat{\eta''}|_\infty \log x/m + |\widehat{\eta''_*}|_\infty}{2\pi^2}(\pi^2-4),$$

which vanishes for q even, and equals

$$\frac{q}{x}\frac{|\widehat{\eta''_*}|_\infty}{2\pi^2}(\pi^2-4)\sum_{\substack{m\le M/q\\ m \text{ odd}}} m + \frac{|\widehat{\eta''}|_\infty}{2\pi^2}(\pi^2-4)\sum_{\substack{m\le M/q\\ m \text{ odd}}} \frac{mq}{x}\log\frac{x}{mq} \tag{11.43}$$

for q odd. To bound the first sum in (11.43), recall that $\sum_{m\le x: m \text{ odd}} m \le (x+1)^2/4$. As for the second sum,

$$\sum_{\substack{m\le M/q\\ m \text{ odd}}} \frac{mq}{x}\log\frac{x}{mq} \le \frac{M}{x}\log\frac{x}{M} + \frac{1}{2}\int_0^{M/q}\frac{tq}{x}\log\frac{x}{tq}dt,$$

where we use the fact that $t \mapsto t\log(x/t)$ is increasing for $t \le x/e$. For the same reason, $(M/x)\log x/M \le 1/e$. Finally,

$$\begin{aligned}\int_0^{M/q}\frac{tq}{x}\log\frac{x}{tq}dt &= -\frac{x}{q}\int_0^{M/x} u\log u du\\ &= -\frac{x}{q}\cdot\frac{1}{2}(M/x)^2\log\frac{M/x}{\sqrt{e}} = \frac{M^2}{2qx}\log\frac{\sqrt{e}x}{M}.\end{aligned}$$

Hence,

$$\frac{|\widehat{\eta''}|_\infty}{2\pi^2}(\pi^2-4)\sum_{\substack{m\le M/q\\ m \text{ odd}}} \frac{mq}{x}\log\frac{x}{mq} \le |\widehat{\eta''}|_\infty\left(\frac{1}{2}-\frac{2}{\pi^2}\right)\left(\frac{1}{e}+\frac{M^2}{4qx}\log\frac{\sqrt{e}x}{M}\right).$$

□

11.4 CYCLING THROUGH MODULI EFFICIENTLY

We must now treat the terms in sums like

$$\sum_m \mu(m)\sum_n e(\alpha mn)\eta\left(\frac{mn}{x}\right) \tag{11.44}$$

coming from m either not divisible by q, or large. "Large" m will arise only if we are working with an approximation $\alpha = a/q + \delta/x$ with $|\delta|$ larger than a constant. In that case, there must be an alternative approximation $\alpha = a'/q' + \delta'/x$, where q' is of reasonable size – larger than q, but smaller than x.

We will then be able to play with the two approximations a/q, a'/q', using one of them to estimate some terms and the other one to estimate the rest. In fact, the alternative approximation a'/q' will be useful precisely for m large.

This is one of the differences between what we are about to do and the classical procedure. Another one is that we will be using our bounds from §11.2. This means not just that we are estimating trigonometric sums rather carefully (so did [DR01]), but that we choose the best estimate to apply in any given situation. Of course, we are also taking advantage of smoothing.

The most basic idea, though, is the same as in Vinogradov's work. As m ranges over an interval of length q, the angles $(a/q)m \bmod 1$ take the values 0, $\pm 1/q$, $\pm 2/q$, ..., each one once. The order is essentially immaterial. Of course, for m small, we (but not Vinogradov) are omitting the case of m divisible by q, i.e., $(a/q)m \bmod 1$ skips the value 0.

11.4.1 A short type I sum: $S_{I,1}$

We begin by bounding a short type I sum, including most terms of the sum $S_{I,1}$. By "short" we mean that the outer sum $\sum_m$ is short enough for αm to be always close to am/q. This simplifies matters greatly. In particular, we will not need to switch approximations from a/q to a'/q' in the manner we described above.

We will also not need an optimal or near-optimal result. What will be essential is to consider the terms with m divisible by q separately. Other than that, our approach will be almost completely classical: we will bound the double sum in (11.45) by splitting the outer sum into intervals of length q, each of which we will bound using a trigonometric-sum estimate – in this case, the estimate (11.12), which is basic and very simple.

Lemma 11.9. *Let $x \geq e$. Let $2\alpha = a/q + \delta/x$, $(a, q) = 1$, $|\delta/x| \leq 1/q^2$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise C^1 function such that $\eta, \eta', \eta'' \in L^1$, $|\eta|_1 = 1$, $\eta(t) = 0$ for $t \leq 0$ and $\eta(t) \geq 0$ for $t > 0$. Let $\eta_*(t) = \eta(t) \log t$ for $t > 0$, $\eta_*(0) = 0$ for $t \leq 0$. Assume that η_* is continuous and that $\eta_*, \eta_*', \eta_*'' \in L^1$.*

Let $M \leq \min(x/2|\delta|q, x)$. Assume $q \leq x$. Then

$$\sum_{\substack{m \leq M \\ m \text{ odd} \\ q \nmid m}} \left| \sum_{n \text{ odd}} (\log n) e(\alpha m n) \eta\left(\frac{mn}{x}\right) \right| \tag{11.45}$$

is at most

$$M \cdot \left(|\eta_*'|_1 + |\eta'|_1 \log \frac{ex}{M} \right) \frac{1}{\pi} \log \frac{7q}{3} + q \cdot (|\eta_*'|_1 + |\eta'|_1 \log x) \frac{1}{\pi} \log \frac{7q}{3}. \tag{11.46}$$

Proof. We will bound the sum $\sum_{m\leq M: q\nmid m} |T_{m,\circ}(\alpha)|$, where

$$T_{m,\circ}(\alpha) = \sum_{n \text{ odd}} (\log n) e(\alpha m n) \eta\left(\frac{mn}{x}\right).$$

We shall split our sum into sums of length at most q, and then bound each one of those sums using Lemma 11.2. Of course, first, we need to estimate $T_{m,\circ}(\alpha)$.

Define $\eta_{(\rho)}$ as in (11.41). Clearly,

$$T_{m,\circ}(\alpha) = \sum_{n \text{ odd}} f_m(n) e(\alpha m \cdot n)$$

for $f_m(t) = \eta_{(x/m)}(mt/x)$. Hence, by (3.23),

$$\begin{aligned} |T_{m,\circ}(\alpha)| &\leq \frac{|f_m'|_1}{2|\sin 2\pi m\alpha|} = \frac{|\eta_{(x/m)}'|_1}{2|\sin 2\pi m\alpha|} \\ &= \frac{|\eta' \log \frac{x}{m} + \eta_*'|_1}{2|\sin 2\pi m\alpha|} \leq \frac{|\eta'|_1 \log \frac{x}{m} + |\eta_*'|_1}{2|\sin 2\pi m\alpha|}. \end{aligned} \tag{11.47}$$

Thus,

$$\begin{aligned} \sum_{\substack{m\leq M \\ q\nmid m}} |T_{m,\circ}(\alpha)| &= \sum_{j=0}^{\lfloor \frac{M-1}{q} \rfloor} \sum_{m=qj+1}^{\min((j+1)q-1, M)} |T_{m,\circ}(\alpha)| \\ &\leq \sum_{j=0}^{\lfloor \frac{M-1}{q} \rfloor} \sum_{m=qj+1}^{\min((j+1)q-1, M)} \frac{|\eta'|_1 \log \frac{x}{m} + |\eta_*'|_1}{2|\sin 2\pi m\alpha|}. \end{aligned}$$

To bound the inner sum, we apply (11.12) with $Q = \lfloor x/|\delta q| \rfloor$ and with 2α instead of α, and obtain

$$\sum_{\substack{m\leq M \\ q\nmid m}} |T_{m,\circ}(\alpha)| \leq \frac{q}{\pi} \log \frac{7q}{3} \cdot \sum_{j=0}^{\lfloor \frac{M-1}{q} \rfloor} \left(|\eta'|_1 \log \frac{x}{qj+1} + |\eta_*'|_1 \right).$$

Now

$$q \sum_{j=0}^{\lfloor M/q \rfloor} |\eta_*'|_1 \leq (M+q)|\eta_*'|_1$$

and

$$q \sum_{j=0}^{\lfloor M/q \rfloor} \log \frac{x}{qj+1} \leq q \log x + \int_0^M \log \frac{x}{t} dt = q \log x + M \log \frac{ex}{M}.$$

Hence, we conclude that

$$\sum_{\substack{m\leq M\\ q\nmid m}} |T_{m,\circ}(\alpha)| \leq \frac{1}{\pi}\left((M+q)|\eta'_*|_1 + \left(M\log\frac{ex}{M} + q\log x\right)|\eta'|_1\right)\log\frac{7q}{3}.$$

□

11.4.2 The basic sum of type I

We will now estimate what we may think of as the most natural type I sum. The treatment will be simpler in some ways than that in the proof of Lemma 11.9, since there will be no factor of $\log n$ in the inner sum. On the other hand, we will no longer assume that the sum is short. This will make it crucial to use an alternative approximation a'/q' to 2α for large values of m if δ is large.

The basic procedure is as follows. We have a double sum (11.48) to estimate; the outer sum is on the variable m. We will have taken out the terms with m small and divisible by q from the beginning; they will be estimated later using the results in §11.3. The terms corresponding to the smallest m ($< q$) get bounded by Lemma 11.1. It remains to see what to do with the remaining m. We will use Lemma 11.5 for m small and switch to the second bound in Lemma 11.4 for all large m. If δ is large, we switch when we must, i.e., when the error in the approximation ma/q to $m\alpha$ becomes large; we then also switch to the approximation a'/q' given by Lemma 2.2. If δ is small, we switch from Lemma 11.5 to Lemma 11.4 roughly at the point at which doing so becomes advantageous.

Lemma 11.10. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ with $2\alpha = a/q + \delta/x$, $(a,q)=1$, $|\delta/x| \leq 1/q^2$. Let $\eta : \mathbb{R}\to\mathbb{R}$ be a continuous, piecewise C^1 function such that $|\eta|_1 = 1$, $\eta', \eta'' \in L^1$ and $\eta(t) = 0$ for $t \leq 0$. Let $c_0 \geq |\widehat{\eta''}|_\infty$, $c \geq \sqrt{c_0}/3\pi$ be given.*

Let $1 \leq D \leq x$. Then, if $|\delta| \leq c$,

$$\sum_{\substack{m\leq D\\ m \text{ odd}\\ q\nmid m \text{ or } m\geq \frac{x}{3|\delta|q}}} \left|\sum_{n \text{ odd}} e(\alpha m n)\eta\left(\frac{mn}{x}\right)\right| \tag{11.48}$$

is at most

$$\frac{\sqrt{c_0 c_1}}{\pi}D + \frac{3c_1}{4}\log^+\frac{3D}{x/cq}\cdot\frac{x}{q} + \left(\frac{\sqrt{c_0c_1}}{\pi}\log^+\frac{D}{q} + \frac{|\eta'|_1}{\pi}\log q + c_2\right)q, \tag{11.49}$$

where

$$c_1 = 1 + \frac{|\eta''|_1}{4}\left(\frac{x}{D}\right)^2, \qquad c_2 = \frac{|\eta'|_1}{\pi}\log\frac{7}{3} + \frac{2\sqrt{3c_0c_1}}{\pi} + \frac{9cc_1}{2} + \frac{2c_0}{\pi^2 c}. \tag{11.50}$$

If $|\delta| \geq c$, the expression in (11.48) is at most

$$\frac{\sqrt{c_0c_1}}{\pi}D + \left(\frac{\sqrt{c_0c_1}}{\pi}\log^+\frac{3D}{x/|\delta|q} + c_3\right)\frac{x}{|\delta|q} + \left(\log^+\frac{3D}{x/|\delta|q} + c_4\right)|\delta|q, \tag{11.51}$$

where

$$c_3 = 2\sqrt{7}\cdot\frac{\sqrt{c_0 c_1}}{\pi}, \qquad c_4 = \frac{9c_1}{2} + \frac{10c_0}{\pi^2 c^2}. \tag{11.52}$$

If $D \leq x/3cq$, then, for $|\delta| \leq c$, the expression in (11.48) is at most

$$\frac{c_0}{3\pi^2 c} D + \left(\frac{2c_0}{\pi^2 c} + \frac{|\eta'|_1}{\pi}\log\frac{7}{3}q\right) q \tag{11.53}$$

if $D \leq x/3|\delta|q$ and $|\delta| \geq c$, the expression in (11.48) is at most

$$\frac{c_0}{3\pi^2 c} D + \frac{10c_0}{\pi^2 c^2}|\delta| q. \tag{11.54}$$

We will take care to make the coefficient of D as small as possible, since the term proportional to D will usually be the main term.

On another issue: as we shall see, we could easily replace $(|\eta'|_1/\pi)\log q$ in (11.49) by $(|\eta'|_1/2)\log^+ x/3cq$, provided that c_2 is simultaneously replaced by

$$c_2' = \frac{|\eta'|_1}{2} + \frac{2\sqrt{3c_0 c_1}}{\pi} + \frac{9cc_1}{2} + \frac{2c_0}{\pi^2 c}. \tag{11.55}$$

Likewise, in (11.53), $(|\eta'|_1/\pi)\log 7q/3$ could be replaced by $(|\eta'|_1/2)\log^+ eD$. The interest in these alternative bounds is that, while we will find the bounds (11.49) and (11.53) as written to be better in our range, the alternative versions would be qualitatively better for q very large. We will discuss why this matter is relevant after the proof.

Proof. Let $Q = x/|\delta|q$, $M = \min(Q/3, D)$. It is our task to bound

$$\sum_{\substack{m\leq M\\ m \text{ odd}, q\nmid m}} |T_m(\alpha)| + \sum_{\substack{M<m\leq D\\ m \text{ odd}}} |T_m(\alpha)|, \tag{11.56}$$

where

$$T_m(\alpha) = \sum_{n \text{ odd}} e(\alpha m n)\eta\left(\frac{mn}{x}\right). \tag{11.57}$$

By (3.11),

$$|T_m(\alpha)| \leq \sum_{n \text{ odd}} \eta\left(\frac{n}{x/m}\right) \leq \frac{x}{2m}|\eta|_1 + \frac{m}{8x}|\eta''|_1. \tag{11.58}$$

At the same time, by (3.23),

$$|T_m(\alpha)| \leq \min\left(\frac{|\eta'|_1}{2|\sin 2\pi m\alpha|}, \frac{m}{x}\frac{c_0}{2|\sin 2\pi m\alpha|^2}\right). \tag{11.59}$$

We can bound the terms $1 \leq m \leq M$ by means of either Lemma 11.4 or Lemma 11.5. A back-of-the envelope calculation suggests that Lemma 11.4 is preferable very roughly when $m > \pi x/\sqrt{c_0}q$; this threshold (or any higher one) is also convenient, since it will allow some terms to cancel.

We will treat the cases of $|\delta| \geq c$ and $|\delta| < c$ separately. The treatment of the latter case will work for $|\delta| \leq c$, and, with some changes, could be made to work for δ arbitrary; we treat $|\delta| \geq c$ separately because we actually want to take advantage of the size of δ for $|\delta| \geq c$.

Note that, since $c \geq \sqrt{c_0}/3\pi$ and $M \leq Q/3 \leq x/3|\delta|q$, we will have that, when $|\delta| \geq c$, all terms $m \leq M$ satisfy $m \leq \pi x/\sqrt{c_0}q$.

Case (a). δ large: $|\delta| \geq c$. Then q cannot be too large, and thus it makes sense to use Lemma 11.5 to bound all terms with $m \leq M$. By (11.59) and Lemma 11.5,

$$\sum_{\substack{m\leq M\\ m \text{ odd}\\ q\nmid m}} |T_m(\alpha)| \leq \sum_{0\leq j<\frac{M}{2q}} \sum_{\substack{m \text{ odd}, q\nmid m\\ 2qj<m<2q(j+1)\\ m\leq M}} \frac{c_0 m/2x}{|\sin 2\pi\alpha m|^2} \leq \frac{c_0}{2x} \sum_{0\leq j<\frac{M}{2q}} 2q(j+1)\cdot\frac{8q^2}{\pi^2}$$
$$= \frac{8c_0q^3}{\pi^2 x}\sum_{j<\frac{M}{2q}+1} j < \frac{4c_0q^3}{\pi^2 x}\left(\frac{M}{2q}+1\right)\left(\frac{M}{2q}+2\right). \tag{11.60}$$

Now, $|\delta| \geq c$ implies both $M \leq Q/3 = x/3|\delta|q \leq x/3cq$ and $c/x \leq |\delta|/x \leq 1/qQ \leq 1/q^2$. Hence, $Mq/x \leq 1/3c$ and $q \leq \sqrt{x/c}$, and we conclude that

$$\sum_{\substack{m\leq M\\ m \text{ odd}, q\nmid m}} |T_m(\alpha)| \leq \frac{c_0}{\pi^2}\cdot\left(\frac{M^2q}{x}+\frac{6Mq^2}{x}+\frac{8q^3}{x}\right) \leq \frac{c_0}{\pi^2}\cdot\left(\frac{M}{3c}+\frac{2q}{c}+\frac{8q}{c}\right). \tag{11.61}$$

If $D \leq Q/3$, then $M = D$ and so (11.61) is all we need: the second sum in (11.56) is empty. Thus we obtain the final bound (11.54).

Assume from now on that $D > Q/3$. To bound the second sum in (11.56), we will use the approximation a'/q' given by Lemma 2.2 instead of a/q. The motivation is the following: if we used the approximation a/q for $m > Q/3$, we would have to count the contribution of all terms with αm very close to 0 (closer than $1/q$) – and that contribution would be too large. When we use a'/q', the contribution of the terms with αm closer to 0 than $1/q'$ is very small: only a fraction $1/q'$ (tiny, since q' is large) of all terms are like that, and the individual contribution of even the largest among them is small, precisely because $m > Q/3$.

Let, then a'/q' be as in Lemma 2.2 (with $\epsilon = |\delta|/x$ and 2α instead of α). Then

$Q/2 < q' \leq Q$. By (11.58), (11.59) and Lemma 11.4,

$$\begin{aligned}
\sum_{\substack{Q/3<m\leq D\\ m \text{ odd}}} |T_m(\alpha)| &\leq \sum_{j=0}^{\infty} \sum_{\substack{m \text{ odd}\\ m>2jq'+\frac{Q}{3}\\ m\leq \min\left(2(j+1)q'+\frac{Q}{3},D\right)}} |T_m(\alpha)| \\
&\leq \sum_{j=0}^{\left\lfloor \frac{D-Q/3}{2q'}\right\rfloor} \sum_{\substack{m \text{ odd}\\ m>2jq'+\frac{Q}{3}\\ m\leq \min\left(2(j+1)q'+\frac{Q}{3},D\right)}} \min\left(\frac{c_1 x}{2m}, \frac{m}{x}\frac{c_0}{2|\sin 2\pi m\alpha|^2}\right) \\
&\leq \sum_{j=0}^{\left\lfloor \frac{D-Q/3}{2q'}\right\rfloor} \left(\frac{3c_1 x/2}{2jq'+Q/3} + \frac{4q'}{\pi}\sqrt{\frac{c_0 c_1}{4}\left(1+\frac{2q'}{2jq'+Q/3}\right)}\right),
\end{aligned}$$

where $c_1 = 1 + |\eta''|_1/4(x/D)^2$. Since $q' > Q/2$,

$$\sum_{j=0}^{\left\lfloor \frac{D-Q/3}{2q'}\right\rfloor} \frac{x}{2jq'+Q/3} \leq \frac{x}{\frac{Q}{3}} + \frac{x}{2q'}\int_{Q/3}^{D}\frac{dt}{t} \leq \frac{3x}{Q} + \frac{x}{Q}\log\frac{D}{Q/3}. \tag{11.62}$$

Since $q' \leq Q$,

$$\begin{aligned}
q'\sum_{j=0}^{\left\lfloor \frac{D-Q/3}{2q'}\right\rfloor} \sqrt{1+\frac{2q'}{2jq'+Q/3}} &\leq q'\sqrt{1+\frac{2Q}{Q/3}} + \frac{1}{2}\int_{Q/3}^{D}\sqrt{1+\frac{2q'}{t}}dt \\
&\leq q'\sqrt{7} + \frac{1}{2}\left(D-\frac{Q}{3}\right) + \frac{q'}{2}\log\frac{D}{Q/3},
\end{aligned} \tag{11.63}$$

since $\sqrt{1+2t} \leq 1+t$ for $t \geq 0$. (Using this simple inequality may seem a little brutal, but it is optimal for $Q/3$ close to D.)

We conclude that $\sum_{Q/3<m\leq D:\, m \text{ odd}} |T_m(\alpha)|$ is at most

$$\begin{aligned}
&\frac{2\sqrt{c_0c_1}}{\pi}\left(\frac{D}{2} + \left(\sqrt{7}-\frac{1}{6}\right)Q\right) \\
&+ \frac{2\sqrt{c_0c_1}}{\pi}\cdot\frac{Q}{2}\log\frac{D}{Q/3} + \frac{3c_1}{2}\left(3+\frac{2}{3}\log^+\frac{D}{Q/3}\right)\frac{x}{Q}.
\end{aligned} \tag{11.64}$$

We sum this to (11.61) (with $M = Q/3$), and obtain that (11.56) is at most

$$\begin{aligned}
&\frac{2\sqrt{c_0c_1}}{\pi}\left(\frac{D}{2} + Q\left(\sqrt{7}+\frac{1}{2}\log\frac{D}{Q/3}\right)\right) \\
&+ \frac{3c_1}{2}\left(3+\frac{2}{3}\log\frac{D}{Q/3}\right)\frac{x}{Q} + \frac{10c_0}{\pi^2 c}q,
\end{aligned} \tag{11.65}$$

where we are using the assumption that $c \geq \sqrt{c_0}/3\pi$, which implies that $(2\sqrt{c_0}/\pi)Q/6$ is at least $c_0 M/3\pi^2 c$. We conclude that (11.51) holds.

Again because $c \geq \sqrt{c_0}/3\pi$, the bound (11.51) is also valid when $D \leq Q/3$: we have proven the bound (11.54), which is much stronger.

Case (b). δ small: $|\delta| \leq c$. Let $D' = \min(x/3cq, D)$. We estimate the first terms of our sum by Lemma 11.2

$$\sum_{\substack{m \leq \min(q,D') \\ q \nmid m}} |T_m(\alpha)| \leq \frac{|\eta'|_1}{2} \sum_{m \leq \min(q,D')} \frac{1}{|\sin 2\pi m\alpha|} \leq \frac{|\eta'|_1}{\pi} q \log^+ \frac{7}{3} q.$$

Alternatively, we could use Lemma 11.1 and obtain the bound $(|\eta'|_1/2) q \log^+ eD'$ instead.

Let us now examine other terms with $m \leq D'$. Since $m \leq D'$ implies $m \leq x/3|\delta|q = Q/3$, we may proceed as in (11.60), and obtain

$$\sum_{\substack{q < m \leq D' \\ q \nmid m, m \text{ odd}}} |T_m(\alpha)| \leq \frac{8c_0 q^3}{\pi^2 x} \sum_{2 \leq j < \frac{D'}{2q}+1} j \leq \frac{4c_0 q^3}{\pi^2 x} \left(\left(\frac{D'}{2q} \right)^2 + 3 \frac{D'}{2q} \right). \tag{11.66}$$

We apply the bounds $D' q^2/x \leq q/3c$ (from $D' \leq x/3cq$) and $(D')^2 q/x \leq D'/3c$ (again from $D' \leq x/3cq$), and conclude that

$$\sum_{\substack{q < m \leq D' \\ q \nmid m, m \text{ odd}}} |T_m(\alpha)| < \frac{4c_0}{\pi^2} \cdot \left(\frac{1}{4} \frac{D'}{3c} + \frac{3}{2} \frac{q}{3c} \right) = \frac{c_0}{3\pi^2 c} (D' + 6q). \tag{11.67}$$

If $x/3cq \geq D$, we stop here. Assume that $x/3cq < D$. We must then estimate the contribution of the terms with $\max(x/3cq, q) < m \leq D$. Let us see how to estimate the contribution of the terms $R < m \leq D$ for $R > 0$ arbitrary. By the second bound in Lemma 11.4:

$$\begin{aligned} \sum_{\substack{R < m \leq D \\ m \text{ odd}}} |T_m(\alpha)| &\leq \sum_{j=0}^{\infty} \sum_{\substack{m \text{ odd} \\ m > 2jq+R \\ m \leq \min(2(j+1)q+R, D)}} |T_m(\alpha)| \\ &\leq \sum_{j=0}^{\lfloor \frac{D-R}{2q} \rfloor} \sum_{\substack{m \text{ odd} \\ m > 2jq+R \\ m \leq \min(2(j+1)q+R, D)}} \min\left(c_1 \frac{x}{2m}, \frac{m}{x} \frac{c_0}{2|\sin 2\pi m\alpha|^2} \right) \\ &\leq \sum_{j=0}^{\lfloor \frac{D-R}{2q} \rfloor} \frac{3c_1 x/2}{2jq+R} + \frac{4q}{\pi} \sqrt{\frac{c_0 c_1}{4} \left(1 + \frac{2q}{2jq+R} \right)}. \end{aligned} \tag{11.68}$$

Note there is no need to use a second approximation a'/q' as in case (a). We are also including all terms with m divisible by q, as we may, since $|T_m(\alpha)|$ is non-negative.

Much as before,

$$\sum_{j=0}^{\lfloor\frac{D-R}{2q}\rfloor} \frac{x}{2jq+R} \leq \frac{x}{R} + \frac{x}{2q}\int_R^D \frac{1}{t}dt \leq \frac{x}{R} + \frac{x}{2q}\log^+\frac{D}{R}, \tag{11.69}$$

and, if $R \geq q$,

$$\begin{aligned}\sum_{j=0}^{\lfloor\frac{D-R}{2q}\rfloor} \sqrt{1+\frac{2q}{2jq+R}} &\leq \sqrt{1+\frac{2q}{R}} + \frac{1}{2q}\int_R^D \sqrt{1+\frac{2q}{t}}dt \\ &\leq \sqrt{3} + \frac{D-R}{2q} + \frac{1}{2}\log^+\frac{D}{q}.\end{aligned} \tag{11.70}$$

Thus, in total, for $R \geq q$,

$$\begin{aligned}\sum_{\substack{R<m\leq D\\ m \text{ odd}}} |T_m(\alpha)| \leq \frac{\sqrt{c_0 c_1}}{\pi}\left(D - R + 2\sqrt{3}\,q + q\log^+\frac{D}{q}\right) \\ + \frac{3c_1}{2}\left(\frac{x}{R} + \frac{x}{2q}\log^+\frac{D}{R}\right)\end{aligned} \tag{11.71}$$

Let us apply this with $R = \max(x/3cq, q)$. Then, by $R \geq D'$, $c \geq \sqrt{c_0}/3\pi$ and $c_1 \geq 1$, the term $-(\sqrt{c_0 c_1}/\pi)R$ is $\leq -(c_0/3\pi^2 c)D'$, and thus eliminates the first term in (11.67). We conclude that

$$\begin{aligned}\sum_{\substack{q<m\leq Q/3\\ q\nmid m, m \text{ odd}}} |T_m(\alpha)| + \sum_{\substack{Q/3<m\leq D\\ m \text{ odd}}} |T_m(\alpha)| \leq \frac{\sqrt{c_0 c_1}}{\pi}\left(D + 2\sqrt{3}q + q\log^+\frac{D}{q}\right) \\ + \frac{3c_1}{2}\frac{x}{2q}\log^+\frac{D}{x/3cq} + \left(\frac{9cc_1}{2} + \frac{2c_0}{\pi^2 c}\right)q.\end{aligned}$$

It is easy to check that this bound is valid even when $x/3cq \geq D$: the first term in (11.67) is then $(c_0/3\pi^2 c)D'/3 = (c_0/3\pi^2 c)D \leq \sqrt{c_0 c_1}D/\pi$ (again by $c \geq \sqrt{c_0}/3\pi$ and $c_1 \geq 1$). □

Let us examine the bound (11.49) given by Lemma 11.10, modified as in the comment right after the statement. As we said, such a modification improves the bound for q very large. In particular, this means that we did not truly need to derive separately a bound that takes advantage of $|\delta| \geq c$; we can simply derive it from our bound for $|\delta| \leq c$.

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ satisfy $2\alpha = a/q + \delta/x$, $(a,q) = 1$, $|\delta/x| \leq 1/qQ_0$, $q \leq Q_0$, where $Q_0 \geq \sqrt{6\pi x/\sqrt{c_0}}$. Then, by Lemma 2.2, there are coprime a', q' with $Q/2 < q' \leq Q$, $Q = x/|\delta|q \geq Q_0$, such that $\delta'/x = \alpha - a'/q'$ satisfies $|\delta'/x| \leq 1/q'Q$. We see that

$|\delta'| < x/q'Q \leq 2x/Q^2 \leq 2x/Q_0^2 \leq \sqrt{c_0}/3\pi \leq c$. Applying (11.49), we obtain that

$$\left| \sum_{\substack{m\leq D \\ m \text{ odd}, q'\nmid m}} \mu(m) \sum_{n \text{ odd}} e(\alpha m n)\eta\left(\frac{mn}{x}\right) \right|$$

is at most

$$\frac{\sqrt{c_0 c_1}}{\pi} D + \left(\frac{\sqrt{c_0 c_1}}{\pi} \log^+ \frac{2D}{x/|\delta|q} + \frac{|\eta'|_1}{2} \log 2|\delta|q + c_2'\right) \frac{x}{|\delta|q} + \frac{3c_1}{2} \log^+ \frac{3cD}{|\delta|q} \cdot |\delta|q, \tag{11.72}$$

since $x/q' \in [x/Q, x/(Q/2)) = [|\delta|q, 2|\delta|q)$. Here c_1 and c_2' are as in (11.50) and (11.55). The term proportional to $|\delta|q$ is minor, since $|\delta|q \leq x/Q_0$. We still have to include the terms divisible by q', but, for the same reason, they contribute very little: by (3.10),

$$\begin{aligned} \sum_{\substack{m\leq D \\ m \text{ odd}, q'|m}} \left| \sum_{n \text{ odd}} \eta\left(\frac{mn}{x}\right) \right| &\leq \sum_{\substack{m\leq D/q' \\ m \text{ odd}}} \left(\frac{x}{2m} + \frac{|\eta'|_1}{2}\right) \\ &\ll \frac{x}{q'} \log^+ \frac{eD}{q'} \leq \frac{2x}{Q_0} \log^+ \frac{2eD}{Q_0}. \end{aligned}$$

How does the bound (11.72) compare to (11.51)? The leading terms are the same. It is easy to show that the term proportional to $x/|\delta|q$ is somewhat larger in (11.72): the sum $9cc_1/2 + 2c_0/\pi^2 c$ in (11.50) is at least $\sqrt{9cc_1 \cdot 4c_0/\pi^2 c} = 6\sqrt{c_0 c_1}/\pi$, and $2\sqrt{3} + 6 > 2\sqrt{7} + \log 3/2$; moreover, there is also the term proportional to $(x/|\delta|q) \log 2|\delta|q$.

It is unsurprising that (11.51) is a little better. We fine-tuned the coefficients of the second-largest terms in (11.49) and (11.51), and cared less about the lesser terms; passing from a/q to a'/q', however, makes the "lesser" terms in (11.49) into the second-largest term in (11.72).

11.5 THE TRIPLE SUM $S_{I,2}$

Now comes the time to focus on another type I sum, namely,

$$\sum_{\substack{v\leq V \\ v \text{ odd}}} \Lambda(v) \sum_{\substack{u\leq U \\ u \text{ odd}}} \mu(u) \sum_{\substack{n \\ n \text{ odd}}} e(\alpha v u n)\eta(vun/x), \tag{11.73}$$

which corresponds to the term $S_{I,2}$ in (10.11). The innermost two sums, on their own, are a sum of type I we have already seen. Accordingly, for q small, we will be able to bound them using Lemma 11.10. If q is large, then that approach does not quite work, since then the approximation av/q to $v\alpha$ is not always good enough. (As we shall later see, we need $q \leq Q/v$ for the approximation to be sufficiently close for our purposes.)

11.5.1 A suboptimal bound on $S_{I,2}$

Fortunately, when q is large, we can also afford to lose a factor of log, since the gains from q will be large. Here is how we shall proceed for q large. First, we set aside the terms in (11.73) with vu divisible by q and no larger than M, where $M \leq x/2|\delta|x$. Then we apply the easy inequality

$$\left|\sum_{v|a} \Lambda(v)\right| \leq \log a \tag{11.74}$$

to obtain that

$$\sum_{\substack{v\leq V\\ v \text{ odd}}} \Lambda(v) \sum_{\substack{u\leq U\\ u \text{ odd}\\ q\nmid uv \text{ or } uv\geq M}} \left|\sum_{\substack{n\\ n \text{ odd}}} e(\alpha vun)\eta(vun/x)\right| \tag{11.75}$$

is at most

$$\sum_{\substack{m\leq UV\\ m \text{ odd}\\ q\nmid m \text{ or } m\geq M}} (\log m) \left|\sum_{\substack{n\\ n \text{ odd}}} e(\alpha mn)\eta(mn/x)\right|. \tag{11.76}$$

We can then simply bound $\log m$ by $\log UV$, and apply Lemma 11.10, our basic bound of type I.

In fact, in practice, we will need only the bound (11.50), and not the bound (11.51), which takes advantage of δ large. This is so because q large implies δ small: if $2\alpha = a/q + \delta/x = a/q + O^*(1/qQ)$, then $|\delta| \leq x/qQ$.

11.5.2 Terms with m divisible by q

It remains to consider the terms with $m \leq M$, m divisible by q. The procedure is in the style of §11.3, with some changes – due in part to our having a triple sum with q large and in part to our aiming at winning only one factor of log, not two. We will obtain some cancellation from the factor $\mu(m)$. The same lemma will work for large q or small q; we will just obtain more cancellation when q is small.

We will first need an easy lemma that will help us with expressions that will arise from our triple sums, here and later.

Lemma 11.11. *Let $q \in \mathbb{Z}^+$. For $n \in \mathbb{Z}^+$, write n_q for $n/(n,q)$. Then, for $x \geq \sqrt{2}$ and $x_1 \geq x_0 \geq 1$,*

$$\sum_{n\leq x} \frac{\Lambda(n)}{n_q} \leq \log \frac{qx}{\sqrt{2}}, \qquad \sum_{x_0<n\leq x_1} \frac{\Lambda(n)}{n_q} \leq \log \frac{qx_1}{x_0} + \frac{\log 3}{3}, \tag{11.77}$$

$$\sum_{x_0<n\leq x_1} \frac{\Lambda(n)}{n_q} \log \frac{x_1}{n} \leq \left(\frac{1}{2}\log\frac{x_1}{x_0} + \log q + \frac{\log 3}{3}\right) \log \frac{x_1}{x_0}. \tag{11.78}$$

Moreover, for $x > 0$,

$$\sum_{n \leq x} \Lambda(n) n(n,q) \leq 0.62008 x^2 + q x \log x. \tag{11.79}$$

Proof. The main observation here is that

$$\sum_{\substack{n \\ (n,q)>1}} \Lambda(n) \frac{(n,q)-1}{n} = \sum_{p|q} (\log p) \left(\sum_{1 \leq j \leq v_p(q)} 1 + \sum_{j > v_p(q)} \frac{p^{v_p(q)}}{p^j} - \sum_j \frac{1}{p^j} \right)$$

$$= \sum_{p|q} (\log p) \left(v_p(q) + \sum_j \frac{1}{p^j} - \sum_j \frac{1}{p^j} \right) = \sum_{p|q} (\log p) v_p(q) = \log q.$$

Hence

$$\sum_{x_0 < n \leq x_1} \frac{\Lambda(n)}{n_q} = \sum_{x_0 < n \leq x_1} \frac{\Lambda(n)}{n} + \sum_{\substack{n \\ (n,q)>1}} \Lambda(n) \frac{(n,q)-1}{n} \leq \log q + \sum_{x_0 < n \leq x_1} \frac{\Lambda(n)}{n}.$$

This is, in particular, true for $x_0 = 0$ and $x_1 = 1$. Similarly,

$$\sum_{x_0 < n \leq x_1} \frac{\Lambda(n)}{n_q} \log \frac{n}{x_0} \leq \log q \log \frac{x_1}{x_0} + \sum_{x_0 < n \leq x_1} \frac{\Lambda(n)}{n} \log \frac{n}{x_0}.$$

We apply Lemma 5.7 (inequalities (5.39), (5.42) and (5.43)) and obtain (11.77) and (11.78).

To obtain (11.79), write

$$\sum_{v \leq x} \Lambda(v) v(v,q) \leq \sum_{v \leq x} \Lambda(v) v + x \sum_{\substack{v \leq x \\ (v,q) \neq 1}} \Lambda(v)(v,q)$$

and apply, on the one hand, Cor. 5.5, and, on the other,

$$\sum_{\substack{v \leq x \\ (v,q) \neq 1}} \Lambda(v)(v,q) \leq \sum_{p|q} (\log p) \sum_{1 \leq j \leq \log_p x} p^{v_p(q)} \leq \sum_{p|q} (\log p) \frac{\log x}{\log p} p^{v_p(q)}$$

$$= (\log x) \sum_{p|q} p^{v_p(q)} \leq q \log x.$$

□

Lemma 11.12. *Let* $\alpha = a/q + \delta/x$, $(a,q) = 1$. *Let* $\eta : \mathbb{R} \to \mathbb{R}$ *be a continuous, piecewise* C^1 *function such that* $\eta, \eta', \eta'' \in L^1$ *and* $\eta(t) = 0$ *for* $t \leq 0$.

Then, for any $U \geq 1$, $V \geq \sqrt{2}$ and $M \leq \min(x/2|\delta|q, UV)$, the absolute value of

$$\sum_{\substack{v\leq V \\ v \text{ odd}}} \sum_{\substack{u\leq U \\ u \text{ odd} \\ uv\leq M, q|uv}} \Lambda(v)\mu(u) \sum_{\substack{n \\ n \text{ odd}}} e(\alpha vun)\eta(vun/x) \tag{11.80}$$

is at most

$$\frac{x}{2q}\left|\widehat{\eta}\left(\frac{-\delta}{2}\right)\right| \log\frac{Vq}{\sqrt{2}} \max_{\substack{r|2q \\ y\geq M/Vq}} m_r(y) + \left(\frac{5}{8}\frac{(U+q)^2V^2}{xq} + \frac{U^2V(\log V)}{x}\right)\cdot c_\eta, \tag{11.81}$$

where $c_\eta = (1/8 - 1/2\pi^2)|\widehat{\eta''}|_\infty$.

As always, m_q is as in (5.45).

Proof. We can assume q is odd, since otherwise (11.80) vanishes. By Lemma 11.7, the expression in (11.80) equals

$$\sum_{\substack{v\leq V \\ v \text{ odd}}} \sum_{\substack{u\leq U \\ u \text{ odd} \\ uv\leq M, q|uv}} \Lambda(v)\mu(u)\left(\frac{\kappa_v x\widehat{\eta}(-\delta/2)}{2vu} + O^*\left(\frac{vu}{x}\frac{|\widehat{\eta''}|_\infty}{2\pi^2}\cdot(\pi^2-4)\right)\right), \tag{11.82}$$

where κ_v is independent of u and satisfies $|\kappa_v| = 1$.

Write $v_q = v/(q,v)$, $q_v = q/(q,v)$. Then

$$\begin{aligned} \sum_{\substack{v\leq V \\ v \text{ odd}}} \sum_{\substack{u\leq U \\ u \text{ odd} \\ uv\leq M, q|uv}} \frac{\Lambda(v)\mu(u)}{vu} &= \sum_{\substack{v\leq V \\ v \text{ odd}}} \frac{\Lambda(v)}{v} \sum_{\substack{u\leq\min(U,M/v) \\ u \text{ odd} \\ q_v|u}} \frac{\mu(u)}{u} \\ &= \sum_{\substack{v\leq V \\ v \text{ odd}}} \frac{\Lambda(v)}{v}\frac{1}{q_v}\cdot m_{2q_v}\left(\min\left(\frac{U}{q_v}, \frac{M}{vq_v}\right)\right). \end{aligned} \tag{11.83}$$

By Lemma 11.11,

$$\sum_{\substack{v\leq V \\ v \text{ odd}}} \frac{\Lambda(v)}{v}\frac{(q,v)}{q} = \frac{1}{q}\sum_{\substack{v\leq V \\ v \text{ odd}}} \frac{\Lambda(v)}{v_q} \leq \frac{1}{q}\log\frac{Vq}{\sqrt{2}}.$$

As for the other terms in (11.82): for any v,

$$\sum_{\substack{u\leq U \\ u \text{ odd} \\ q|uv}} u = \sum_{\substack{u\leq U \\ u \text{ odd} \\ q_v|u}} u = q_v \sum_{\substack{u\leq \frac{U}{q_v} \\ u \text{ odd}}} u \leq \frac{q_v}{4}\left(\frac{U}{q_v}+1\right)^2 = \frac{1}{4}\left(\frac{U^2}{q_v} + 2U + q_v\right).$$

Hence

$$\sum_{\substack{v\leq V\\ v \text{ odd}}}\sum_{\substack{u\leq U\\ u \text{ odd}\\ q|uv}} \Lambda(v)uv \leq \frac{1}{4}\frac{U^2}{q}\sum_{\substack{v\leq V\\ v \text{ odd}}}\Lambda(v)v(v,q) + \frac{2U+q}{4}\sum_{\substack{v\leq V\\ v \text{ odd}}}\Lambda(v)v.$$

By Cor. 5.5 and (11.79), this is at most

$$\frac{1}{4}\left(\frac{U^2}{q}+2U+q\right)\cdot 0.62008V^2 + \frac{U^2}{4q}\cdot qV\log V,$$

which is at most $0.62008((U+q)V)^2/4q + U^2V(\log V)/4$.

□

11.5.3 A bound on $S_{I,2}$ for q small

Let us now give an estimate for $q \leq Q/V$ small. It will be based on Lemma 11.10. It will have to be more delicate than the bound in the previous subsection, since, for q small, we cannot afford to waste factors of log.

Lemma 11.13. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ with $2\alpha = a/q + \delta/x$, $(a,q) = 1$, $|\delta/x| \leq 1/qQ$, $q \leq Q$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise C^1 function such that $|\eta|_1 = 1$, $\eta', \eta'' \in L^1$ and $\eta(t) = 0$ for $t \leq 0$. Let $c_0 = |\widehat{\eta''}|_\infty$. Let $c \geq \sqrt{c_0}/3\pi$ be given. Let $U, V \geq 1$. Assume that $Q \leq UV/e$ and $q \leq Q/V$.*

Then

$$\sum_{\substack{v\leq V\\ v \text{ odd}}}\Lambda(v)\sum_{\substack{u\leq U\\ u \text{ odd}\\ q\nmid uv \text{ or } uv\geq \frac{x}{3|\delta|q}}}\left|\sum_{\substack{n\\ n \text{ odd}}} e(\alpha vun)\eta(vun/x)\right| \tag{11.84}$$

is at most

$$c'\frac{\sqrt{c_0c_1}}{\pi}UV + \frac{3c_1}{4}L\left(\frac{L}{2}+\log 3^{1/3}q\right)\frac{x}{q} + \left(\frac{\sqrt{c_0c_1}}{\pi}\log\frac{U}{q} + \frac{|\eta'|_1\log q}{\pi} + c_2\right)c'Q, \tag{11.85}$$

where $L = \log^+(3cqUV/x)$, $c' = \psi(V)/V$ and

$$c_1 = 1 + \frac{|\eta''|_1}{4}\left(\frac{x}{UV}\right)^2, \qquad c_2 = \frac{|\eta'|_1}{\pi}\log\frac{7}{3} + \frac{2\sqrt{3c_0c_1}}{\pi} + \frac{9cc_1}{2} + \frac{2c_0}{\pi^2c}. \tag{11.86}$$

If $|\delta| \geq c$, the expression in (11.84) is at most

$$c'\frac{\sqrt{c_0c_1}}{\pi}UV + \frac{\sqrt{c_0c_1}}{\pi}\left(L'\left(\frac{L'}{2}+\log 3^{1/3}q\right) + c_3(L'+\log 3^{1/3}q)\right)\frac{x}{|\delta|q} + (L'+c_4)\frac{c'x}{Q/V}, \tag{11.87}$$

where $L' = \log^+(3|\delta|qUV/x)$, c_1 and c' are as above and c_3, c_4 are as in (11.52).

Here, as usual, $\psi(V) = \sum_{v \le V} \Lambda(v)$.

Proof. Since $q \le Q/V$, for any $v \le V$,

$$2v\alpha = \frac{va}{q} + O^*\left(\frac{v}{Qq}\right) = \frac{va}{q} + O^*\left(\frac{1}{q^2}\right),$$

i.e., va/q is a valid approximation to $2v\alpha$. Moreover, for $Q_v = Q/v$, we see that $2v\alpha = (va/q) + O^*(1/qQ_v)$. We can write

$$2v\alpha = \frac{va}{q} + \frac{v\delta}{x} = \frac{va}{q} + \frac{\delta}{x/v},$$

that is, δ does not change. We can also write va/q as $(av_q)/q_v$, where $v_q = v/\gcd(v,q)$ and $q_v = q/\gcd(v,q)$; note that av_q and q_v are coprime.

Of course, the expression $e(\alpha vun)\eta(vun/x)$ inside the double sum in (11.84) can be rewritten as $e((v\alpha)\cdot un)\eta(un/(x/v))$.

We can clearly apply Lemma 11.10 to bound the inner sum in (11.84); we have just verified that its conditions hold. We obtain that, if $|\delta| < c$, the value of (11.84) is at most

$$\begin{aligned}&\sum_{v\le V} \Lambda(v)\left(\frac{\sqrt{c_0c_1}}{\pi}U + \frac{3c_1}{4}\frac{x}{vq_v}\log^+\frac{3U}{x/cvq_v}\right)\\ &+ \sum_{v\le V}\Lambda(v)\left(\frac{\sqrt{c_0c_1}}{\pi}\log^+\frac{U}{q_v} + \frac{|\eta'|_1}{\pi}\log q_v + c_2\right)q_v,\end{aligned} \tag{11.88}$$

where $q_v = q/(q,v)$, $c_1 = 1 + |\eta''|_1/4(x/UV)^2$ and c_2 is as in (11.50). If $|\delta| \ge c$, the value of (11.84) is at most

$$\begin{aligned}&\sum_{v\le V}\Lambda(v)\frac{\sqrt{c_0c_1}}{\pi}\left(U + \frac{x}{|\delta|vq_v}\log^+\frac{3U}{\frac{x/v}{|\delta|q_v}}\right) + \sum_{\frac{x/|\delta|q}{3U}<v\le V}\Lambda(v)\frac{x}{|\delta|vq_v}c_3\\ &+ \sum_{v\le V}\Lambda(v)\left(\log^+\frac{3U}{\frac{x/v}{|\delta|q_v}} + c_4\right)|\delta|q_v,\end{aligned} \tag{11.89}$$

where c_3 and c_4 are as in (11.52).

The second sum in (11.89) deserves comment. We can restrict the range of v in this way because $v \le x/3U|\delta|q$ implies $U \le (x/v)/3|\delta|q \le (x/v)/3|\delta|q_v$, which is the condition for the bound (11.54) to be valid. That bound (11.54) would contribute a term of the form $c_0U/3c\pi^2$, which, by the assumption $c \ge \sqrt{c_0}/3\pi$, is no larger than the term $\sqrt{c_0c_1}U/\pi$ already present in the first line of (11.89).

As will become clear at the end of the proof, we need to restrict the range of v in this way so as to avoid a term of size $(\log V)/|\delta|q$; such a term would give something of size $(\log x)/|\delta|q$ in the final bound on exponential sums, and we cannot afford that.

To estimate the second term in the first line of (11.88) or of (11.89), we apply (11.78):

$$\begin{aligned}\sum_{v\le V}\frac{\Lambda(v)}{vq_v}\log^+\frac{3U}{x/cvq_v} &\le \frac{1}{q}\sum_{v\le V}\frac{\Lambda(v)}{v_q}\log^+\frac{v}{x/3cUq}\\ &\le \frac{1}{q}\left(\frac{L}{2}+\log q+\frac{\log 3}{3}\right)L,\end{aligned} \tag{11.90}$$

where $L=\log^+(V/(x/3cUq))$. The same holds with $|\delta|\neq 0$ instead of c.

Recall that we are assuming that $Q\le UV/e$ and $q\le Q/V$. Hence $q\le U/e$ (implying that $q_v\log^+(U/q_v)\le q\log(U/q)$) and $Vq\le Q$. Using (11.90), we see that (11.88) is at most

$$\begin{aligned}&\frac{\sqrt{c_0c_1}}{\pi}U\sum_{v\le V}\Lambda(v)+\frac{3c_1}{4}\frac{x}{q}L\left(\frac{L}{2}+\log 3^{1/3}q\right)\\ &+\left(\frac{\sqrt{c_0c_1}}{\pi}\log\frac{U}{q}+\frac{|\eta'|_1}{\pi}\log q+c_2\right)\cdot\frac{Q}{V}\sum_{v\le V}\Lambda(v).\end{aligned} \tag{11.91}$$

The expressions in (11.89) get estimated similarly. In particular,

$$\sum_{v\le V}\Lambda(v)\log^+\frac{3U}{\frac{x/v}{|\delta|q_v}}\le L'\sum_{v\le V}\Lambda(v), \tag{11.92}$$

where $L'=\log^+ 3UV|\delta|q/x$. By (11.77),

$$\sum_{\frac{x}{3U|\delta|q}<v\le V}\frac{\Lambda(v)}{vq_v}=\frac{1}{q}\sum_{\frac{x}{3U|\delta|q}<v\le V}\frac{\Lambda(v)}{v_q}\le\frac{1}{q}\left(L'+\log q+\frac{\log 3}{3}\right)$$

thanks to (5.42), provided, of course, that $x/3U|\delta|q<V$; if that is not the case, the sum is empty.

We conclude that the expression in (11.89) is at most

$$\begin{aligned}&\frac{\sqrt{c_0c_1}}{\pi}\left(U\psi(V)+\frac{x}{|\delta|q}\left(L'\left(\frac{L'}{2}+\log 3^{1/3}q\right)+c_3(L'+\log 3^{1/3}q)\right)\right)\\ &+|\delta|q\,(L'+c_4)\,\psi(V).\end{aligned}$$

Note, lastly, that $|\delta|q\le x/Q$, simply because $|\delta|/x\le 1/qQ$. □

As elsewhere, there is some overlapping between the condition $uv>x/3|\delta|q$ in (11.80) and the condition $uv\le x/2|\delta|q$ in (11.84). Because there is an absolute value around the inner sum in (11.84), this overlap is harmless. The reason for having a 3 rather than a 2 in one instance is just that the condition $y_2\le Q/3$ was used in Lemma 11.5 instead of $y_2\le Q/2$ so as to make the bounds from that lemma good enough to fit nicely with others.