Kähler groups and the geometry of Kähler manifolds

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Introduction

This text is an introduction to the topic of Kähler groups. The main problem is to determine which finitely presentable groups appear to be fundamental groups of compact Kähler manifolds. Such groups are called Kähler groups.

A general answer to this question seems to be out of reach for the moment but several restrictions are known, notably on the cohomology of these groups. Moreover some constructions of complicated Kähler groups are known, which may explain the difficulty of the problem.

I have tried to make this text as self-contained as possible. The only prerequisite is some familiarity with differential and Riemannian geometries, elementary algebraic topology and cohomology theories.

An important part of this text follows the book [ABC+96] which is the reference on the subject. and which also deals with several fundamental techniques which I cannot develop here.

Part I consists in some preliminary results needed in the rest of the text. These are results in complex geometry, in algebraic topology and in Kähler geometry.

Part II contains partial results of the general question stated above. I also explain why this problem is interesting. Most results are stated without proof since they involve some techniques which are beyond the scope of the text.

Part III is the heart of the text. I collect several results concerning fibrations of compact Kähler manifolds over complex curves and then concentrate on such manifolds with 1-dimensional Albanese image. Following [Kot11], this study leads notably to a proof of a result first showed in [DS09], saying that Kähler groups which are also 3-manifolds groups are finite.
Part I

Preliminaries
Chapter 1

Complex geometry

In this report, we are primarily interested in a certain class of complex manifolds: the compact Kähler manifolds. By sake of self-containedness, we include this chapter which deals with basic results in complex geometry. The first two sections treat only the theory of complex manifolds whereas the third and the fourth are respectively about complex analytic geometry and complex algebraic geometry.

Any textbook dealing with complex manifolds treats the material in the first two sections, we recommend [Huy04], [Wel07] or [Voi08]. The classical reference for complex analytic geometry is [GR84]. For algebraic geometry, the first chapters of [Sha94] or [Har77] are sufficient.

1.1 Complex manifolds

Definition 1.1. A complex manifold of dimension $n$ is a differentiable manifold of dimension $2n$ for which the transition maps are holomorphic with respect to a fixed isomorphism $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. It is easily seen that the notion of holomorphic maps between complex manifolds makes sense.

The underlying differentiable manifold $X$ of a complex manifold $M$ carries an almost complex structure, that is an endomorphism $I$ of the (real) tangent bundle $TX$ such that $I^2 = -1$. One can then consider the complexified tangent bundle $\mathcal{T}_C X := TX \otimes_{\mathbb{R}} \mathbb{C}$ with its natural decomposition

$$\mathcal{T}_C X = \mathcal{T}^{1,0} M \oplus \mathcal{T}^{0,1} M,$$

where $\mathcal{T}^{1,0} X$ and $\mathcal{T}^{0,1} X$ are the eigenspaces of $I \otimes id$ for the eigenvalue $i$ and $-i$.

This induces bidegree decompositions of the bundles of differentiable complex-valued forms $\Lambda^k_C(X)$ and of their sheaf of sections $\mathcal{A}^k_C$:

$$\Lambda^k_C X = \bigoplus_{p+q=k} \Lambda^{p,q} M;$$
$$\mathcal{A}^k_C = \bigoplus_{p+q=k} \mathcal{A}^{p,q}. \quad (1.2)$$

If $z_1, \ldots, z_n$ are complex local coordinates for the complex manifold $M$ then natural real coordinates of the underlying differentiable manifold $X$ are defined by $x_j = \Re z_j$ and $y_j = \Im z_j$. Local sections of $\mathcal{T}_C X$ are then given by

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right);$$
$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

3
and the $\frac{\partial}{\partial z_j}$'s (resp. the $\frac{\partial}{\partial \bar{z}_j}$'s), $1 \leq j \leq n$, form a basis of $T^{1,0}M$ (resp. $T^{0,1}M$).

Denote by $dz_j = dx_j + idy_j$ and by $d\bar{z}_j = dx_j - idy_j$ the dual 1-forms. A section of $\Lambda^{p,q}M$ can locally be written as

$$\sum_{|I|=p,|J|=q} f_{I,J} dz_I \wedge d\bar{z}_J,$$

where the $f_{I,J}$ are complex-valued differential functions, $I$ and $J$ are subsets of $\{1, \ldots, n\}$ and if $I = \{i_1, \ldots, i_p\}$ with $i_1 < \cdots < i_p$ then $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ (and the same convention for $d\bar{z}_J$).

Let $\alpha = f dz_I \wedge d\bar{z}_J$ be a form in $\Lambda^{p,q}(M)$. We define differential operators $\partial$ and $\bar{\partial}$ on $X$ by

$$\partial \alpha = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J;$$

$$\bar{\partial} \alpha = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J.$$

These definitions do not depend on the particular local coordinates $z_1, \ldots, z_n$. The operators $\partial$ and $\bar{\partial}$ satisfy the corresponding Leibniz's rules. In fact $\partial \alpha$ (resp. $\bar{\partial} \alpha$) is just the part of type $(p+1, q)$ (resp. $(p, q+1)$) of $d\alpha$ if $\alpha$ is of type $(p, q)$. In particular $d$, $\partial$ and $\bar{\partial}$ are related by the equality $d = \partial + \bar{\partial}$ which implies $\partial^2 = \partial \bar{\partial} + \bar{\partial} \partial = \bar{\partial}^2 = 0$ by taking the bidegree decomposition.

**Definition 1.2.** The $p$-Dolbeault complex of the complex manifold $M^n$ with underlying differentiable manifold $X^{2n}$ is the complex

$$\Lambda^{p,0}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Lambda^{p,n}(M) \xrightarrow{\bar{\partial}} 0.$$ 

The $(p, q)$-Dolbeault cohomology of $M$ is the $q$th cohomology group of this complex. It is denoted by $H^{p,q}(M)$.

One checks easily that the kernel of $\bar{\partial} : \Lambda^{p,0}(M) \rightarrow \Lambda^{p,1}(M)$ is the sheaf $\Omega^p(M)$ of holomorphic $p$-forms on $M$, i.e. the forms which can locally be written as $\sum_{|I|=p} f_I dz_I$ where the $f_I$ are holomorphic functions in $z_1, \ldots, z_n$. Since the sheaves $\Lambda^{p,q}(M)$ are acyclic, the abstract de Rham theorem (see [Vois08], chapter 4, or the Appendix of [Huy04]) gives the following:

**Theorem 1.1.** $H^{p,q}(M) = H^q(M, \Omega^p)$.

One can naturally ask if the decomposition (1.2) at the level of forms is still valid at the level of cohomology. This is false in general but works for the class of Kähler manifolds which we will study later. For the moment we generalise the Dolbeault cohomology for holomorphic bundles over complex manifolds.

**Definition 1.3.** A holomorphic vector bundle $E$ over a complex manifold $M$ is a complex vector bundle over its underlying differentiable manifold $X$ such that the transition functions are holomorphic with respect to the holomorphic structure on $X$.

There is a natural extension $\bar{\partial}_E$ of the differential operator $\bar{\partial}$ to sections of a holomorphic vector bundle $E$. Locally a section of the sheaf $\Lambda^{p,q}(M) \otimes E$ can be identified with $k$ sections of $\Lambda^{p,q}(M)$ where $k$ is the complex rank of $E$. The operator $\bar{\partial}_E$ is defined locally by applying $\bar{\partial}$ on each coordinate. The holomorphicity of $E$ implies that this definition is independant of the local coordinates we choose. Hence we obtain a global operator $\bar{\partial}_E : \Lambda^{p,q}(E) \rightarrow \Lambda^{p,q+1}(E)$.

The operator $\bar{\partial}_E$ verifies the same properties than $\bar{\partial}$. In particular there is a natural definition of a Dolbeault complex and we denote by $H^{p,q}(M, E)$ the corresponding groups of cohomology. As above, one has the equality $H^{p,q}(M, E) = H^q(M, \Omega^p \otimes E)$ since $\Omega^p \otimes E$ is the kernel of $\bar{\partial}_E : \Lambda^{p,0}(E) \rightarrow \Lambda^{p,1}(E)$.
1.2 Hodge Theory on a complex manifold

Let $(X^n, g)$ be a compact Riemannian oriented (differentiable) manifold. The metric $g$ induces metrics on the bundles $\Lambda^k(X)$ of differentiable forms of degree $k$. If $(e_1, \ldots, e_n)$ is an orthonormal oriented basis of $\mathcal{T}_xX$ for some $x \in X$ then we define $\text{vol}_x = e^1 \wedge \ldots \wedge e^n$ in $\Lambda^n_2X$. This gives in fact a volume form on $X$, i.e. a non-vanishing section $\text{vol}$ of $\Lambda^nX$.

One can use $\text{vol}$ to define an operator $*$ on $\Lambda^*_cX$ by the implicit formula $\alpha \wedge * \beta = g(\alpha, \beta) \text{vol}$. The $*$-operator is an involution (up to the sign) and sends $\Lambda^kX$ to $\Lambda^{n-k}X$. We can use $*$ to construct new operators on $X$. For instance, on $\Lambda^k_c(X)$ the operator $d^*$ is defined by

$$d^* = (-1)^{n(k+1)} * d *.$$

The differential operators $d$ of degree 1 and $d^*$ of degree $-1$ give birth to a Laplacian operator of degree 0:

$$\Delta = dd^* + d^*d.$$

Finally, for forms in $\Lambda^k_c(X)$ the scalar product $(, )$ is defined by

$$(\alpha, \beta) = \int_X g(\alpha, \beta) \text{vol} = \int_X \alpha \wedge \beta.$$

By the Stokes formula and the sign of $*$, one obtains the equalities $(d\alpha, \beta) = (\alpha, d^* \beta)$ and $(\Delta \alpha, \beta) = (\alpha, \Delta \beta)$.

**Definition 1.4.** A form $\alpha \in \Lambda^k_c(X)$ is harmonic if $\Delta \alpha = 0$ or equivalently if $d\alpha = d^* \alpha = 0$.

The space of harmonic $k$-forms is denoted by $\mathcal{H}^k_c(X)$.

The following theorem is the main result of Hodge theory.

**Theorem 1.2 (Hodge decomposition).** Let $(X, g)$ be a compact Riemannian oriented manifold. Then there is an orthogonal decomposition for $(, )$

$$\Lambda^k_c(X) = d(\Lambda^{k-1}_c(X)) \oplus \mathcal{H}^k_c(X) \oplus d^*(\Lambda^{k+1}_c(X)).$$

Moreover the spaces $\mathcal{H}^k_c(X)$ are finite-dimensional.

Let us give a generalisation of this for complex manifolds. For the sake of simplicity, we won’t distinguish anymore between the complex manifold $M$ and its underlying differentiable manifold $X$, denoting them both by $X$.

**Definition 1.5.** A complex hermitian manifold $X$ is a complex manifold endowed with a Riemannian metric $g$ compatible with the almost complex structure $I$: $g(I(v), I(w)) = g(v, w)$, for every $v, w \in \mathcal{T}X$.

Then $X$ is a compact Riemannian oriented (by $I$) manifold. Starting from $\partial$ and $\bar{\partial}$, we define differential operators $\partial^*$ and $\bar{\partial}^*$ similarly to what was done for $d^*$. Moreover there are associated Laplacians: $\Delta_d = dd^* + d^*d$, $\Delta_{\bar{\partial}} = \partial\partial^* + \partial^*\partial$ and $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$.

If $E$ is a holomorphic vector bundle endowed with a hermitian metric, we also have a formal adjoint $\bar{\partial}^*_E$ and a Laplacian $\Delta_E = \bar{\partial}\bar{\partial}^*_E + \bar{\partial}^*\bar{\partial}_E$.

We denote by $\mathcal{H}_d(X)$, $\mathcal{H}_{\partial}(X)$, $\mathcal{H}_{\bar{\partial}}(X)$ and $\mathcal{H}_{\bar{\partial}^*_E}(X, E)$ the corresponding spaces of harmonic forms. These spaces depend on the metric $g$ (and on the hermitian metric on $E$ for $\mathcal{H}_{\bar{\partial}^*_E}(X, E)$).

We have the following Hodge decompositions.
CHAPTER 1. COMPLEX GEOMETRY

**Theorem 1.3** (Hodge decomposition on a compact hermitian manifold). Let \((X, g)\) be a compact hermitian manifold. Then there are two orthogonal decompositions

\[
A^{p,q}(X) = \partial (A^{p-1,q}(X)) \oplus \bar{\partial}^* (\bar{A}^{p+1,q}(X)), \\
\bar{A}^{p,q}(X) = \bar{\partial} (\bar{A}^{p-1,q}(X)) \oplus \partial^* (A^{p+1,q}(X)).
\]

If \(E\) is a holomorphic vector bundle on \(M\) endowed with a hermitian metric then

\[
A^{p,q}(X, E) = \bar{\partial}_E (A^{p,q-1}(X)) \oplus \bar{\partial}_E (A^{p,q-1}(X)) \oplus \bar{\partial}_E (\bar{A}^{p,q+1}(X, E)).
\]

Moreover all the spaces of harmonic forms are finite-dimensional.

Since a harmonic form for \(\bar{\partial}\) is \(\bar{\partial}\)-closed, we have a natural morphism \(\mathcal{H}^{p,q}(X) \to H^{p,q}(X)\). One sees easily that this morphism is injective. The Hodge decomposition implies bijectivity. So any class in \(H^{p,q}(X)\) can be uniquely represented by a harmonic form. There is a similar statement for \(H^{p,q}(X, E)\).

The idea of Hodge theory is to study the properties of the cohomology classes by the properties of the corresponding harmonic maps. An example of this is the Serre duality.

**Theorem 1.4** (Serre duality). If \(E\) is a holomorphic hermitian vector bundle on a compact hermitian manifold \(M\), there is a \(\mathbb{C}\)-antilinear isomorphism

\[
H^{p,q}(X, E) \approx H^{n-p,n-q}(X, E^*).
\]

In order to prove this one uses the \(\bar{\ast}_E\)-operator

\[
\bar{\ast}_E : \Lambda^{p,q} \otimes E \to \Lambda^{n-p,n-q} \otimes E^*
\]

which is the \(*\)-operator followed by complex conjugation on the level of forms and the \(\mathbb{C}\)-antilinear isomorphism from \(E\) to its dual induced by its hermitian metric. This \(\bar{\ast}_E\)-operator is an involution (up to the sign) and it sends \(\mathcal{H}^{p,q}(X, E)\) to \(\mathcal{H}^{n-p,n-q}(X, E^*)\). This shows that Serre duality is true at the level of harmonic forms, hence it is true at the level of cohomology.

**Remark 1.1.**

- It is not hard to prove that there exists in fact a natural \(\mathbb{C}\)-linear isomorphism

\[
H^{p,q}(X, E) = H^{n-p,n-q}(X, E^*)^*\]

which does not depend on the two metrics, contrary to the isomorphism induced by \(\bar{\ast}_E\).

- Since \(H^{p,q}(X, E) = H^q(X, \Omega^p \otimes E)\), Serre duality implies

\[
H^p(X, E) = H^{n-p}(X, \Omega^n \otimes E^*)^*\]

1.3 Complex analytic geometry

In the two preceding sections, we considered only complex manifolds, hence smooth objects. Nevertheless, we will sometimes need to use the more general theory of complex spaces, where singularities can appear. This is a very rich subject, hence we can only give a brief presentations of the techniques used and some important statements without proofs. We refer to [GR84] for an introduction to the subject.
1.3. **COMPLEX ANALYTIC GEOMETRY**

**Complex spaces**

A complex manifold is a space which locally looks like a domain in $\mathbb{C}^n$. We want to define a bigger class of spaces, whose local behaviours are more complicated.

Like in algebraic geometry, one can first define interesting subspaces of domains in $\mathbb{C}^n$, namely the zero sets of a finite number of holomorphic functions.

**Definition 1.6.** A complex model space $(Y, \mathcal{O}_Y)$ is a ringed space such that

- $Y$ is in a domain $D$ of some $\mathbb{C}^n$ and is the support of the sheaf of rings $\mathcal{O}_D/\mathcal{I}$, where $\mathcal{I}$ is an ideal sheaf of finite type in $\mathcal{O}_D$;
- $\mathcal{O}_Y := (\mathcal{O}_D/\mathcal{I})|_Y$.

A complex space is essentially a space which locally looks like a complex model space.

**Definition 1.7.** A complex space $(X, \mathcal{O}_X)$ is a $\mathbb{C}$-ringed space such that $X$ is Hausdorff and every point $p$ in $X$ has an open neighbourhood $U$ for which the induced $\mathbb{C}$-ringed structure $(U, \mathcal{O}_U)$ is isomorphic (as $\mathbb{C}$-ringed spaces) to a complex model space $(Y, \mathcal{O}_Y)$.

**Remark 1.2.**

- If $D$ is a domain in some $\mathbb{C}^n$, the ringed space $(D, \mathcal{O}_D)$ is a model space. A complex space is a complex manifold if and only if it is locally isomorphic to ringed spaces of this form.
- Then notion of holomorphic maps between complex spaces makes sense, by mean of the sheaves of rings on the spaces. In this way, we can define the category of complex spaces and the category of complex manifolds is a full subcategory of that category.
- We will only deal with reduced complex spaces $(X, \mathcal{O}_X)$ whose rings of germs $\mathcal{O}_{X,x}$ are without nilpotent elements, for all $x$ in $X$. We will not precise this anymore in the following.

**Analytic sets**

Analytic sets are locally zero sets of holomorphic functions.

**Definition 1.8.** Let $(X, \mathcal{O}_X)$ be a complex space. A subspace $Y \subset X$ is analytic if for every $x \in X$, there exists a neighbourhood $U$ of $x$ in $X$ and holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}_X(U)$ such that

$$Y \cap U = \{x \in U \mid f_1(x) = \cdots = f_k(x) = 0\}$$

**Remark 1.3.**

- This is a not obvious theorem that every analytic set can naturally be given the structure of a reduced closed complex space.
- It is not always clear from the definition that some subspace in a complex space is an analytic set. The theory of coherent sheaves is an important tool for proving this kind of things. We will only need the proper mapping theorem.

**Theorem 1.5** (Proper mapping theorem, [GR84], p. 213). Let $f$ be a proper holomorphic map between complex spaces $X$ and $Y$. Then $f(X)$ is analytic in $Y$.

Let $A$ be the analytic set $\{z_1z_2 = 0\}$ in $\mathbb{C}^2$. One can see $A$ as the union of two lines intersecting in 0. This example motivates the following definition.
Definition 1.9. Let $A$ be an analytic set in a complex space $X$. One says that $A$ is irreducible if $A$ cannot be written as an union of two proper analytic sets.

The following theorem is more difficult than the corresponding local statement.

Theorem 1.6 (Global decomposition in irreducible components). With the same notations, $A$ decomposes as the union of finitely many irreducible analytic sets.

Dimension of complex spaces, integration over an analytic set

The notion of dimension is more difficult to manipulate than in the case of manifolds because of the singularities. The general theory is developed in [GR84]; we just give what we need.

Proposition 1.7. Let $X$ be an irreducible complex space. Then the set $S(X)$ of singularities of $X$ (points where $X$ is not smooth) is a proper analytic set in $X$ and $X^* = X - S(X)$ is a connected complex manifold.

Definition 1.10. The dimension of an irreducible complex space $X$ is the complex dimension of $X^*$. A (possibly reducible) complex space $X$ has dimension $k$ if all its irreducible components have dimension $k$.

Proposition 1.8. If $A$ is a nowhere dense analytic set in a complex space of dimension $k$, then the dimension of $A$ is at most $k - 1$.

We will not go through the details of integration over an analytic set $V$ in a compact complex manifold $X$ (see [GH94], p. 31–33 and 60–61). One just defines the integral over an analytic set $V$ as the integral over the smooth locus $V^*$ and: ([GH94], p. 33)

Integration over analytic subvarieties is much the same as integration over submanifolds.

Remark 1.4. In particular, one can dually define the fundamental class of $V$, which is a class in $H_{2k}(X)$, where $k$ is the dimension of $V$.

Normal spaces and Stein factorization

In a normal space, there can be singularities but there are not too bad.

Definition 1.11. A complex space $(X, \mathcal{O}_X)$ is a normal space if for every $x \in X$, the ring $\mathcal{O}_{X,x}$ is integrally closed.

Remark 1.5.

- A complex manifold is a normal space.
- The converse is not true but in a normal space $X$, the analytic set $S(X)$ of singularities is of codimension at least 2 (see [GR84], p. 128–129). For a general reduced complex space, this codimension is only bigger than 1.

The following theorem - obvious consequence of the second remark - will be of great use:

Theorem 1.9 (Characteization of normal curves). A complex space of dimension 1 is normal if and only if it is a complex manifold, i.e. a Riemann surface.

When a complex space $(X, \mathcal{O}_X)$ has too many singularities, one can try to desingularize it.
1.4. COMPLEX ALGEBRAIC GEOMETRY

**Definition 1.12.** Let $X$ be a reduced complex space. A *normalization* $\xi : \tilde{X} \to X$ is the data of a normal space $\tilde{X}$ and a holomorphic map $\xi : \tilde{X} \to X$ such that:

- There exists a subset $T$ in $X$ which is thin in $X$, i.e. contained in an analytic set of $X$ of codimension at least 1, and whose inverse image $\xi^{-1}(T)$ is thin in $\tilde{X}$;
- The restriction map $\xi : \tilde{X} \setminus \xi^{-1}(T) \to X \setminus T$ is biholomorphic.

**Theorem 1.10** (Existence of a normalization, [GR84], p. 161). Every reduced complex space $X$ has a normalization $\xi : \tilde{X} \to X$.

We also state an easy consequence of the property of lifting 8.4.3 in [GR84].

**Proposition 1.11.** Let $f$ be a holomorphic map from a complex manifold of positive dimension $X$ to a reduced curve $C$. Then $f$ can be uniquely lifted to the normalization of $C$, i.e. there exists a unique holomorphic map $\bar{f} : X \to \bar{C}$ such that $f = \xi \circ \bar{f}$, where $\xi : C \to \bar{C}$ is the normalization map.

The last theorem will be used several times desingularize the situation:

**Theorem 1.12** (Stein factorization, [GR84], p. 213). Let $f$ be a proper holomorphic map from $X$ to $Y$. Then $f$ admis a unique (up to isomorphism) factorization

$$X \xrightarrow{\hat{f}} \hat{Y} \xrightarrow{g} Y$$

such that

- $\hat{f}$ is a proper and holomorphic surjection; $g$ is finite (closed and each fibre finite) and holomorphic; $f = g \circ \hat{f}$;
- All fibers of $\hat{f}$ are connected.

Moreover, if $X$ is normal then $\hat{Y}$ is normal too.

**Remark 1.6.** The following particular case will be the most important for us: $X$ is a compact complex manifold, $Y$ is a (possibly singular) curve. Then $\hat{Y}$ is normal since $X$ is normal and is a curve since $Y$ is a curve and $g$ is finite. Hence it is a smooth curve.

### 1.4 Complex algebraic geometry

We will need a few results of algebraic geometry in the proof of Serre theorem. We recall them here and refer to [Har77].

The Segre embedding

Let $m, s$ be two fixed integers and consider the product $X = \mathbb{C}P^s \times \cdots \times \mathbb{C}P^s$ with $m$ factors. We can embed $X$ in some $\mathbb{C}P^N$ by the Segre embedding

$$\iota : (x_1, \ldots, x_m) \mapsto [x_{01} \cdots x_{0m} : \cdots : x_{i_1} \cdots x_{im} : \cdots : x_{s_1} \cdots x_{sm}],$$

where $x_{ji}$ is the $j$-th homogeneous coordinate of $x_i \in \mathbb{C}P^s$ and $i_1, \ldots, i_m$ evolve in all $m$-tuples of integers between 0 and $s$. With easy arguments we obtain

- $N = (s + 1)^m - 1$;
- In the analytic viewpoint, $\iota$ is well defined and is a smooth embedding of $X$ into $\mathbb{C}P^N$;
- In the algebraic viewpoint, the image $\iota(X)$ is a projective variety in $\mathbb{C}P^N$. 
CHAPTER 1. COMPLEX GEOMETRY

Bertini theorems

The following theorem of Bertini shows that the section of a smooth projective subvariety with a generic hypeplane is smooth again. It will allow us to decrease the dimension or to get rid of the singularities.

**Theorem 1.13** (Bertini theorem, [Har77], p. 179). Let $X$ be a smooth connected projective variety in $\mathbb{C}P^n$. Then the generic hyperplane $H$ in $\mathbb{C}P^n$ is such that:

- $H \cap X$ is a (possibly disconnected) smooth projective variety;
- The dimension of $H \cap X$ is the dimension of $X$ minus 1.

**Corollary 1.14.** Let $Y \subset X$ be two closed projective varieties in $\mathbb{C}P^n$ such that $X \setminus Y$ is smooth. If $d$ is strictly bigger than the dimension of $Y$, the generic linear subspace $L$ of codimension $d$ in $\mathbb{C}P^n$ is such that:

- $L$ does not meet $Y$;
- $L \cap X$ is a smooth projective variety.
Chapter 2

Algebraic topology

In this chapter, we recall some constructions of algebraic topology. Our aim is to define the classifying space $BG$ of a group $G$ and to relate the low-degree cohomologies of $BG$ and $X$, where $X$ is any CW-complex having $G$ as fundamental group. References for this chapter are [Hat01] and [Whi78].

2.1 Homotopy groups

Definition 2.1. Let $n$ be a nonnegative integer and let $X$ be a topological space. We fix a base-point $*$ on each sphere $S^n$. Then for $x \in X$, $\pi_n(X, x)$ is the set of homotopy classes (rel $*$) of continuous maps $(S^n, *) \to (X, x)$. We call $\pi_1(X, x)$ the fundamental group of $X$ at the point $x$.

Remark 2.1. Some facts about these objects:

- The set $\pi_0(X, x)$ is the set of pathwise connected components of $X$, with a privileged element, corresponding to the component where $x$ lies;
- For $n \geq 1$ the sets $\pi_n(X, x)$ are naturally endowed with a group structure;
- For $n \geq 2$ these groups are abelian;
- Every continuous map $f : X \to Y$ induces group homomorphisms $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$. Moreover $f_*$ depends only on the homotopy class of $f$;
- If $X$ is path connected then $\pi_n(X, x)$ is non-canonically isomorphic to $\pi_n(X, y)$, for every $x$ and $y$ in $X$;
- When the base-points are not important, we will simply omit them;

Definition 2.2. A path-connected space $X$ is aspherical if for all $n \geq 2$, $\pi_n(X) = 0$.

The following proposition is fundamental to understand all constructions involving CW-complexes and homotopy groups.

Proposition 2.1. Let $f$ be a map from $S^n$ to $X$. Then $f$ can be extended to a map $\tilde{f} : D^{n+1} \to X$ if and only if it is a constant map up to homotopy.

Proof. Let $F$ be an homotopy from $f$ to the constant map $c_x$ with $x \in X$. One has $F(y, 0) = f(y)$, $F(y, 1) = x$ and $F(*, t) = x$ for all $y \in S^n$ and all $t \in I$. The ball $D^{n+1}$ can be seen as the quotient $S^n \times I / \sim$ where $(y, 1) \sim (z, 1)$ for all $y, z \in S^n$ and the inclusion $S^n \subset D^{n+1}$.
corresponds to the map \( S^n \to S^n \times \{0\} \). With this identification \( F \) descends to a map \( \tilde{f} : D^{n+1} \to X \) which extends \( f \).

Conversely let \( \tilde{f} \) be an extension of \( f \) to \( D^{n+1} \). Since \( D^{n+1} \) is contractible there exists an homotopy (rel \( * \)) from the inclusion \((S^n, *) \subset (D^{n+1}, *)\) to the constant map to \( * \), i.e. there exists a map : \( G : S^n \times I \to D^{n+1} \) such that \( G(*,t) = * \), \( G(y,0) = y \) and \( G(y,1) = * \). Then \( \tilde{f} \circ G \) is a homotopy from \( f \) to a constant map. \( \square \)

**Corollary 2.2.** Let \( f : X \to Y \) be a continuous map to an aspherical space \( Y \). Add a \( p \)-cell to \( X \) by an attaching map \( \alpha : S^{p-1} \to X \) and denote by \( W \) the space \( X \cup_A D^p \). If \( p \geq 3 \) then \( f \) extends to \( W \).

**Proof.** The map \( f \circ \alpha : S^{p-1} \to Y \) extends to \( D^p \) since \( \pi_{p-1}(Y) = \{0\} \). This gives the desired extension of \( f \). \( \square \)

### 2.2 Eilenberg-Mac Lane spaces

**Definition 2.3.** Let \( n \) be a positive integer and let \( G \) be a group, abelian if \( n > 1 \). An *Eilenberg-Mac Lane* \( K(G,n) \) is a CW-complex whose homotopy groups are trivial, except \( \pi_n(K(G,n)) \) which is isomorphic to \( G \). A *classifying space* for \( G \), denoted by \( BG \) is a space \( K(G,1) \).

We need to recall a useful property about approximation by cellular maps.

**Definition 2.4.** A *cellular map* between CW-complexes is a map which sends \( k \)-skeleton into \( k \)-skeleton for every \( k \).

**Theorem 2.3** (Approximation by cellular maps, [Whi78], p.77–78). Let \( X,Y \) be CW-complexes and let \( A \) be a subcomplex of \( X \). Then every continuous map between \( X \) and \( Y \), whose restriction to \( A \) is cellular, is homotopic (rel \( A \)) to a cellular map.

**Corollary 2.4.** Let \((X,x)\) be a CW-complex with base-point \( x \) in the 0-skeleton.

- The natural map \( \pi_n(X^{(n)},x) \to \pi_n(X,x) \) is surjective;
- \( \pi_n(X,x) = \pi_n(X^{(n+1)},x) \).

**Proof.** The first point only means that every map from the CW-complex \((S^n, \ast)\) of dimension \( n \) to \((X,x)\) is homotopic (rel \( \ast \)) to a map from \((S^n, \ast)\) to \((X^{(n)},x)\) which is a direct application of the theorem with \( A = \ast \).

For the second point one has to remark that \( S^n \times I \) can be naturally endowed with a structure of CW-complex of dimension \( n + 1 \) and that \( A = \{\ast\} \times I \cup S^n \times \partial I \) is a subcomplex of \( S^n \times I \). The natural map \( \pi_n(X^{(n+1)},x) \to \pi_n(X,x) \) is surjective by the first point. For the injectivity consider a class \([f]\) in the kernel and represent it by a cellular map \( f \) from \((S^n, \ast)\) to \((X^{(n)},x)\). Since \([f]\) is in the kernel, \( f \) is homotopic in \( X \) to the constant map to \( x \). So there is a continuous map \( F \) from \( S^n \times I \) to \( X \) such that \( F(y,0) = f(y), F(y,1) = x \) and \( F(*,t) = x \). Applying the theorem with the subcomplex \( A \subset X \) this gives another homotopy which verify the previous equalities but with values in \( X^{(n+1)} \). So \([f]\) is zero in \( \pi_n(X^{(n+1)},x) \). \( \square \)

**Theorem 2.5** (Existence of Eilenberg-Mac Lane spaces). *Eilenberg-Mac Lane spaces* \( K(G,n) \) exist for every \( n \) and every group \( G \), abelian if \( n > 1 \).

**Proof.** We distinguish between the cases \( n = 1 \) and \( n > 1 \).
2.3. Properties of classifying spaces

- If $n = 1$, take for $G$ a presentation $G = \langle a_i \rangle$. Consider $X^{(1)}$, a wedge of circles indexed by $I$ and meeting at some point $y \in X^{(1)}$. Denote these circles by $(S^1, y)$ and identify them to the circle with base point $(S^1, *)$. By some applications $a_i$ from $(S^1, *)$ to $(Y^{(1)}, y)$ which are homeomorphisms onto their image $(S^1, y)$.

Since the relations $r_j$ can be written as words in the $a_i$ they define by concatenation maps $r_j$ from $(S^1, *)$ to $(Y^{(1)}, y)$. Considered as attaching maps they define a CW-complex $Y^{(2)}$ with 1-skeleton $Y^{(1)}$ and 2-cells indexed by $J$. The fundamental group of $Y$ is a free group in $p$ letters and by construction the Van Kampen theorem implies that the fundamental group of $Y^{(2)}$ is $G$.

Consider generators $c_k$ for $\pi_2(Y^{(2)})$ and representative maps $S^2 \to Y^{(2)}$ still written $c_k$. Define $Y^{(3)}$ to be the CW-complex with 2-skeleton $Y^{(2)}$ and 3-cells corresponding to the attaching maps $c_k$. By the second point of the corollary $\pi_1(Y^{(3)}) = G$. By the first point, $\pi_2(Y^{(3)}) = 0$ since the generators of $\pi_2(Y^{(2)})$ are null-homotopic in $Y^{(3)}$ by construction, using proposition 2.1.

Continuing this construction one obtains a (possibly infinite-dimensional) CW-complex $Y$ with the desired properties.

- For $n > 1$, one sees $G$ as a quotient of a free abelian group and does the same, beginning with a wedge of spheres of dimension $n$.

\[\Box\]

2.3 Properties of classifying spaces

Since we are mostly interested in the fundamental group we will now restrict our attention to classifying spaces.

**Theorem 2.6.** Let $G$ be a group. Then there is a unique classifying space $BG$ up to homotopy equivalence.

**Proof.** Let $X$ be any $BG$ and $Y$ be the $K(G, 1)$ constructed in the proof above with a presentation $G = \langle a_i \rangle$. Fix an isomorphism $\pi_1(X, x) = G$ and denote by $y$ the base-point of $Y$ which is the common point of the circles in the 1-skeleton. Represent $a_i$ by an application $a_i : (S^1, *) \to (X, x)$. Then one has a natural continuous map $\alpha$ from $(Y^{(1)}, y)$ to $(X, x)$ which is defined to be $a_i$ on the circle in $Y$ corresponding to $a_i$.

This application extends to the 2-skeleton of $Y$ if and only if the composition of $\alpha$ and the attaching maps of the 2-cells of $Y$ are null-homotopic in $\pi_1(X, x)$. Since by construction they can be represented by the relations $b_j$ which are zero in $G$ this point is true and we still denote by $\alpha$ this extension to the 2-skeleton. Note that $\alpha_*$ is by construction the identity of $G$.

Using the asphericity of $X$ and corollary 2.2 one can extend $\alpha$ to a map from $(Y, y)$ to $(X, x)$ which is an isomorphism on all homotopy groups (since they are zero for $n > 1$). Thus it is a weak homotopy equivalence, hence a homotopy equivalence by the Whitehead theorem (see [Whi78], p. 220–221).

**Remark 2.2.** In particular one can simply write $BG$ for a classifying space of $G$ when only properties invariant up to homotopy equivalence are dealt with. The main application of this is to define the cohomology algebra of a group $G$ as the cohomology algebra of its classifying space (over any ground ring).
Proof. The construction is essentially the same as the previous one. One uses the models constructed above for $BG$ and $BH$. Then the homomorphism $\phi$ naturally induces a map from the 1-skeleton of $BG$ to the 1-skeleton of $BH$. Since $\phi$ is a group homomorphism, $\phi(0) = 0$ and so an attaching map to the 1-skeleton of $BG$ (which corresponds to a relation in $G$) induces an attaching map to the 1-skeleton of $BH$ which is null-homotopic in $BH$. Hence the application extends to the 2-skeleton and then to all of $BG$ because $BH$ is aspherical. By construction, $B\phi_\ast = \phi$. 

Remark 2.3. In fact this map is unique up to homotopy. Hence one can speak of the map induced by $\phi$ in group (co)homology: this is $B\phi_\ast$ or $B\phi^\ast$.

Theorem 2.8 (Classifying map of a CW-complex). Let $X$ be a CW-complex. Then there exists a continuous map $c_X$ from $X$ to $B\pi_1(X)$ which is an isomorphism on $\pi_1$. Moreover it is an isomorphism on $H^1$ and a monomorphism on $H^2$ for any coefficient ring. We call $c_X$ a classifying map of $X$.

Proof. If a group $G$ is already given as the fundamental group of some CW-complex $X$, one has another construction for its classifying space: $\pi_2(X)$ can be represented by maps from $S^2$ to $X^{(2)}$ so we attach 3-cells to $X^{(2)}$ to kill $\pi_2(X)$ and we continue possibly infinitely many times to kill all $\pi_n(X)$ for $n \geq 2$ by attaching $(n + 1)$-cells to the $n$-skeleton of the new space we have just constructed. As in the proof of the existence of Eilenberg-MacLane spaces, this space is a classifying space for $\pi_1(X)$.

With this construction $X$ can be seen as a subcomplex of $B\pi_1(X)$ and we denote by $c_X$ the inclusion of $X$ in $B\pi_1(X)$ with this model. Then $c_X$ is the identity on $\pi_1$ since we only attach cells of dimension at least 3. Hence it is an isomorphism on $H_1(\cdot; \mathbb{Z})$ since $H_1(Y; \mathbb{Z})$ is the abelianization of $\pi_1(Y)$ for any path-connected space $Y$. By the universal coefficient theorem for cohomology and the five lemma (see [Hat01]), $c_X^\ast$ is also an isomorphism on $H^1(\cdot; \mathbb{A})$, for any coefficient ring $\mathbb{A}$.

The following lemma proves that it is also a monomorphism on $H^2$.

Lemma 2.9. Let $n \geq 2$ be an integer. Let $X$ be a CW-complex and suppose that $Y$ is obtained by adjoining cells of dimension at least $n + 1$ to $X$. Then the natural morphism $H^n(Y) \to H^n(X)$ is a monomorphism.

Proof. Suppose that we add only one cell: $Y = X \cup e^p$ for some $p \geq n + 1$ and some attaching map $\phi$. Then denote by $U$ the open set $Y$ minus the center of the ball $e^p$ and by $V$ the open ball with this center and of radius half the radius of $e^p$. Then $Y = U \cup V$, $X$ is a retract by deformation of $U$, $V$ is contractible and $U \cap V$ retracts to a sphere $S^{p-1}$. By the Mayer-Vietoris theorem in cohomology (see [Hat01]) we have an exact sequence

$$\ldots \to H^{n-1}(U \cap V) \to H^n(Y) \to H^n(U) \oplus H^n(V) \to \ldots$$

which becomes

$$\ldots \to H^{n-1}(S^{p-1}) \to H^n(Y) \to H^n(X) \to \ldots,$$

since $V$ is contractible and $n > 0$. Because $1 < n < p$, $H^{n-1}(S^{p-1})$ is zero, which shows that the map $H^n(Y) \to H^n(X)$ (which is induced by the inclusion of $X$) is a monomorphism.

If there is a finite number of cells, one can do an induction to prove the claim. Finally the general case is obtained by an inductive limit argument for cohomology.
Chapter 3

Kähler geometry and the Albanese torus

Kähler geometry is the study of an important class of complex hermitian manifolds $M$ whose associated 2-form is symplectic. This implies a remarkable bidegree decomposition of the cohomology of $M$ with complex coefficients. To every compact Kähler manifold $X$, one can attach a complex torus which carries some geometrical properties of $X$. A general reference for this chapter is [Huy04].

3.1 Kähler manifolds

Definition 3.1. If $X$ is a complex hermitian manifold with Riemannian metric $g$ and almost complex structure $I$, its associated 2-form is $\omega(v, w) = g(I(v), w)$. It is a real form of type $(1, 1)$.

Definition 3.2. A Kähler manifold is a complex manifold for which there exists a hermitian structure whose associated 2-form $\omega$ is $d$-closed, i.e. symplectic. In this case we say that $\omega$ is a Kähler form.

Example 3.1.

- The projective space $\mathbb{C}P^n$ is Kähler with the Fubini-Study metric.
- A submanifold of a Kähler manifold is Kähler by restriction of the metric.

The Kähler condition on a complex hermitian manifold implies a great rigidity between the different operators that one can consider in such a manifold. This is expressed by the so-called Kähler identities which imply the equality of Laplacians.

Theorem 3.1 (Equality of Laplacians). On a complex manifold endowed with a Kähler metric, $\Delta_\bar{\partial} = \Delta = 1/2\Delta_d$.

In particular the spaces of harmonic forms $H^p_q(X, g)$ and $H^p_q(X, g)$ are equal.

Corollary 3.2. On a compact Kähler manifold $(X, g)$ there exists an isomorphism

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$
Remark 3.1.

- In fact this decomposition does not depend on the metric $g$, in the following sense: if $\alpha$ (resp. $\alpha'$) is a harmonic form of type $(p, q)$ with respect to a Kähler metric $g$ (resp. $g'$) and if $\alpha$ and $\alpha'$ represent the same element in $H^{p,q}(X)$, then they represent the same element in $H^{p+q}(X, \mathbb{C})$.

- Note that the equalities $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and $H^{p,q}(X) = H^{n-p,n-q}(X)^{\ast}$ (see the remark (1.2)) imply some restrictions on the cohomology of $X$. For instance it is straightforward to see that the odd Betti numbers of $X$ have to be even.

We finally state three useful results where the Kähler assumption is essential. The first two propositions are consequences of the Hard Lefschetz theorem (see [Huy04] or [GH94]); the third one shows that a surjective holomorphic map from a compact Kähler manifold is always injective in cohomology. We take the proof from [Cam93].

**Proposition 3.3.** Let $\alpha$ be a non-zero element in $H^{p,0}(X)$ (i.e. a global holomorphic $p$-form) in a compact Kähler manifold $X$. Then $\alpha \wedge \overline{\alpha} \in H^{p,p}(X)$ is non-zero.

**Proposition 3.4.** Let $(X, \omega)$ be a compact Kähler manifold of complex dimension $n$. Then for all $k \leq n$, the bilinear pairing

$$H^k(X, \mathbb{C}) \times H^k(X, \mathbb{C}) \to \mathbb{C}$$

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta \wedge \omega^{n-k}$$

is non-degenerate.

**Proposition 3.5.** Let $f$ be a holomorphic surjective map from a compact Kähler manifold $X$ to a complex manifold $Y$. Then $f$ is injective in cohomology.

**Proof.** Let $\omega$ be a Kähler form on $X$. Denote by $n$ (resp. $m$) the complex dimension of $X$ (resp. $Y$). Since $f$ is surjective, $n \geq m$. The set $Z$ of critical values of $f$ in $Y$ is an analytic set by the proper mapping theorem and has no interior by Sard’s lemma. We consider the open dense set $Y^* = Y - Z$ and $X^* = f^{-1}(Y^*)$. Over $Y^*$, $f$ is submersive, hence locally a trivial fibration (see theorem 7.1). For $F$ a fiber of $f$ in $X^*$, we consider $k_F = \int_F \omega^{n-m}$ which is positive since $F$ is a compact complex submanifold of $X$. We claim that this quantity is independent of the fiber $F$. Indeed, since $f$ is locally trivial, two closed fibers $F$ and $F'$ are the boundary of some submanifold with boundary of $X$ and the closedness of $\omega$ shows that $k_F = k_F'$ by the Stokes formula. Moreover, the analytic set $Z$ cannot disconnect $Y$ (see [GR84]), hence the quantity $k_F$ is the same for every fiber over a point $y$ in $Y^*$.

If $\alpha$ is a $2m$-form in $Y$, then

$$\int_X f^* (\alpha) \wedge \omega^{n-m} = \int_{X^*} f^* (\alpha) \wedge \omega^{n-m} \text{ by density}$$

$$= k \times \int_{Y^*} \alpha \text{ by a calculus, remembering that the fibration is locally trivial}$$

$$= k \times \int_Y \alpha \text{ by density.}$$
Consider $\beta$ in $H^i(Y)$ and let $\gamma$ be arbitrary in $H^{2n-i}(Y)$. If $f^*(\beta) = 0$ in $H^i(X)$, then the above calculus shows that $k \times f_Y^* \beta \wedge \gamma = 0$. By Poincaré duality, this implies $\beta = 0$ and we get the injectivity of $f^*$.

### 3.2 Construction of the Albanese torus and the Albanese map

Let $X$ be a compact Kähler manifold of complex dimension $n$. By the universal coefficient theorem

$$H^{2n-1}(X, \mathbb{R}) = H^{2n-1}(X, \mathbb{Z}) \otimes \mathbb{R}.$$  

So $H^{2n-1}(X, \mathbb{Z})$ modulo the torsion is a lattice in $H^{2n-1}(X, \mathbb{R})$. One also has an inclusion of $H^{2n-1}(X, \mathbb{R})$ in $H^{2n-1}(X, \mathbb{C}) = H^{n,n-1}(X) \oplus H^{n-1,n}(X)$. Since $H^{n-1,n}(X) = H^{n,n-1}(X)$, the intersection of the lattice with both summands is zero. So by projecting this lattice on $H^{n-1,n}(X)$ one can see $H^{2n-1}(X, \mathbb{Z})$ modulo torsion as a lattice in $H^{n-1,n}(X)$.

**Definition 3.3.** The Albanese torus of $X$ is defined by

$$\text{Alb}(X) = \frac{H^{n-1,n}(X)}{\text{im}(H^{2n-1}(X, \mathbb{Z}))},$$

where $\text{im}(H^{2n-1}(X, \mathbb{Z}))$ is the lattice in $H^{n-1,n}(X)$, and we endow $\text{Alb}(X)$ with its natural complex structure.

In order to interpret this torus in a simpler way, one uses Poincaré and Serre dualities (see [Hat01] or [BT82] for Poincaré duality). We have natural isomorphisms $H^{2n-1}(X, \mathbb{Z}) = H_1(X, \mathbb{Z})$, $H^{2n-1}(X, \mathbb{C}) = H_1(X, \mathbb{C})$ and $H^{n-1,n}(X) = H^{1,0}(X)^* = H^0(X, \Omega_X)^*$. Moreover these isomorphisms are compatible in the sense that we have a commutative diagram

$$\begin{align*}
H^{n-1,n}(X) &\overset{\iota}{\longrightarrow} H^{2n-1}(X, \mathbb{C}) \\
\downarrow S &\quad \downarrow P \\
H^{1,0}(X)^* &\overset{r}{\longleftarrow} H^1(X, \mathbb{C})^*
\end{align*}$$

where $\iota$ is the inclusion and $r$ the restriction of linear forms. In order to see this one uses the definition of the isomorphism involved in Serre duality; (see [Huy04], p. 168 – 171).

The Albanese torus can thus be identified with the quotient

$$\text{Alb}(X) \cong \frac{H^0(X, \Omega_X)^*}{\rho(H_1(X, Z))},$$

where $\rho$ is given by

$$\rho : H_1(X, \mathbb{Z}) \to H^0(X, \Omega_X)^*$$

$$[\gamma] \mapsto ([\alpha] \mapsto \int_\gamma \alpha).$$

There exists a holomorphic map from $X$ to $\text{Alb}(X)$. If one chooses a point $p_0$ in $X$, then it is given by

$$\text{alb}_X : X \to \text{Alb}(X)$$

$$p \mapsto ([\alpha] \mapsto \int_{p_0}^p \alpha).$$
This map is well-defined since by taking the values in \( \text{Alb}(X) \) we do not have to pay attention to the particular path between \( p_0 \) and \( p \). Moreover it is holomorphic: it suffices to consider the differential of the function \( p \mapsto \int_{p_0}^{p} \alpha \) for a holomorphic 1-form \( \alpha \). Since the fundamental theorem of calculus says exactly that this differential at \( p \) is \( \alpha(p) \) this proves that \( \text{alb}_X \) is holomorphic and that its differential is:

\[
d_p \text{alb}_X(Z) = (\alpha \mapsto \alpha(p)(Z)). \tag{3.1}
\]

### 3.3 Properties of the Albanese map

**Theorem 3.6.**

- \( \text{alb}_X^* \) is an isomorphism between \( H^0(\text{Alb}(X), \Omega_{\text{Alb}_X}) \) and \( H^0(X, \Omega_X) \).

- For every holomorphic map \( f : X \to T \) from \( X \) to a torus, there exists a unique holomorphic map \( g \) from \( \text{Alb}(X) \) to \( T \) such that \( f = g \circ \text{alb}_X \).

**Proof.**

- For every torus of the form \( T = V/L \) where \( V \) is a complex vector space and \( L \) a lattice in it, \( H^0(T, \Omega_T) \) can naturally be identified with \( V^* \) since the tangent bundle is trivial. If \( \alpha_1, \ldots, \alpha_p \) is a basis for \( H^0(X, \Omega_X)^* \) then its dual basis \( \alpha^1, \ldots, \alpha^p \) is a basis for \( H^0(\text{Alb}(X), \Omega_{\text{Alb}_X}) \). With this identification and by the formula (3.1) the pullback of holomorphic 1-forms is the identity of \( H^0(X, \Omega_X) \). This proves the first point.

- Since the pullback of a holomorphic map sends holomorphic 1-forms to holomorphic 1-forms and by naturality, we have a commutative diagram

\[
\begin{array}{ccc}
H_1(X, \mathbb{Z}) & \longrightarrow & H^0(X, \Omega_X)^* \\
\downarrow f^* & & \downarrow f^* \\
H_1(T, \mathbb{Z})^* & \longrightarrow & H^0(T, \Omega_T)^*
\end{array}
\]

This proves that \( f_* \) on \( H^0(X, \Omega_X)^* \) induces a map \( g \) from \( \text{Alb}(X) \) to \( \text{Alb}(T) \). By naturality in the definition of the Albanese map, we then have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{alb}_X} & \text{Alb}(X) \\
\downarrow f & & \downarrow g \\
T & \xrightarrow{\text{alb}_T} & \text{Alb}(T)
\end{array}
\]

when we choose for the base-point on \( T \) the image by \( f \) of the base-point on \( X \). It is straightforward to see that \( \text{Alb}(T) \) can be identified to \( T \) and the Albanese map \( \text{alb}_T \) is just a translation on \( T \). Hence changing \( g \) by a translation gives the commutative diagram we looked for.

For the uniqueness consider \( g_1 \) and \( g_2 \) two solutions of the problem. Then their difference \( g \) is a holomorphic map from \( \text{Alb}(X) \) to \( T \) which vanishes on \( \text{alb}_X(X) \), hence in \( 0 \), image of the base-point of \( X \). By the following lemma (3.7), \( g \) is a group homomorphism and so, if we prove that \( \text{alb}_X(X) \) is not contained in any proper subtorus of \( \text{Alb}(X) \), this will show that \( g = 0 \). Since the cohomology of \( X \) comes from the pullback of \( f \) corestricted to its image in \( \text{Alb}(X) \), the first point of the theorem proves this. This concludes the proof.
Lemma 3.7. Let \( f : T_1 \to T_2 \) be a holomorphic map between two tori. Then \( f \) is a group homomorphism followed by a translation.

**Proof.** Without loss of generality we can assume that \( f(0) = 0 \). The differential of \( f \) lives in the vector bundle \( T^*T_1 \otimes f^*TT_2 \) over \( T_1 \). This bundle is trivial and \( T_1 \) is compact so \( df \) has to be constant. Writing \( T_1 = \mathbb{C}^n/\Gamma_1 \) and \( T_2 = \mathbb{C}^m/\Gamma_2 \) one can identify \( TT^1 \) with \( T^1 \times \mathbb{C}^n \) and \( TT^2 \) with \( T^2 \times \mathbb{C}^m \). Then \( df \) is given by a single \( \mathbb{C} \)-linear map \( A : \mathbb{C}^n \to \mathbb{C}^m \). Now it is sufficient to show that \( f \) comes from the map \( A \).

Denote by \( \tilde{f} \) the unique map \( \mathbb{C}^n \to \mathbb{C}^m \) which lifts \( f \circ \pi_1 : \mathbb{C}^n \to T_2 \) and sends \( 0 \) to \( 0 \). This map is holomorphic since it is locally the map \( f \) in some charts and we want to prove that \( A = \tilde{f} \). By the identity theorem we can do this only in neighbourhoods of \( 0 \). Locally one can identify \( f \) and \( \tilde{f} \) and \( df \) and \( d\tilde{f} \) since the differentials of the projection maps are the identities with the identifications done above. If \( z \) is in a little neighbourhood of \( 0 \in \mathbb{C}^n \) and \( \gamma \) is a smooth path from \( 0 \) to \( z \), one has

\[
\tilde{f}(z) = \int_0^1 df(\gamma(t)) dt = \int_0^1 A(\gamma'(t)) dt = A(\gamma(1)) - A(\gamma(0))
\]

\( \tilde{f}(z) = A(z) \).

\( \square \)

The complex dimension of the Albanese torus of \( X \) is \( h^{1,0}(X) = \frac{b_1(X)}{2} \). We are particularly interested in the dimension of the image of \( X \) by the Albanese map. Note that by the proper mapping theorem (see 1.5), this image is an analytic subset of \( \text{Alb}(X) \) and by 1.4, one can speak of its fundamental class in homology.

**Definition 3.4.** Let \( X \) be a complex manifold. The **cup-length of \( X \) for real cohomology** is the maximum \( l \) such that the natural morphism \( \Lambda^lH^1(X, \mathbb{R}) \to H^l(X, \mathbb{R}) \) is not identically zero. Similarly the **cup-length of \( X \) for holomorphic forms** is the maximum \( l \) such that the natural morphism \( \Lambda^lH^{1,0}(X) \to H^{l,0}(X) \) is not identically zero.

**Remark 3.2.** The first definition can be made for any topological space \( X \) and any ground field and is independant on the ground field of characteristic 0.

**Definition 3.5.** We denote by \( a(X) \) the dimension of the (possibly singular) Albanese image of \( X \), and call it the **Albanese dimension of \( X \)**.

**Theorem 3.8.** The Albanese dimension of \( X \) is the cup-length of \( X \) for holomorphic forms, which is also half the cup-length of \( X \) for real cohomology.

**Proof.**

- First we show the equality between the cup-length for holomorphic forms and half the cup-length for real cohomology. Suppose that there exist \( \alpha_1, \ldots, \alpha_k \) holomorphic 1-forms such that \( \alpha_1 \wedge \ldots \wedge \alpha_k \neq 0 \). Then by corollary 3.3, \( \alpha_1 \wedge \ldots \wedge \alpha_k \wedge \alpha_1 \wedge \ldots \wedge \alpha_k \neq 0 \) in \( H^1(X, \mathbb{C}) = H^1(X, \mathbb{R}) \otimes \mathbb{C} \). Developing this with real and imaginary parts, one sees that at least one of the \( 2^k \) wedges of \( 2k \) forms in \( H^1(X, \mathbb{R}) \) has to be non-zero. This proves
that the cup-length for real cohomology is at least twice the cup-length for holomorphic forms.

Conversely suppose that every wedge of \( k + 1 \) forms in \( H^{1,0}(X) \) is zero and consider \( \alpha_1, \ldots, \alpha_{2k+1} \) in \( H^1(X, \mathbb{R}) \). If \( \alpha_1 \wedge \ldots \wedge \alpha_{2k+1} \) were not zero then by developing in \((0,1)\) and \((1,0)\) parts one obtains \( \beta_1 \wedge \ldots \wedge \beta_{2k+1} \neq 0 \) for some \( \beta_i \) either in \( H^{1,0}(X) \) or in \( H^{0,1}(X) \). But this wedge of \( 2k + 1 \) elements contains at least \( k + 1 \) elements in \( H^{1,0}(X) \) or in \( H^{0,1}(X) \). Both cases contradict the assumption since for the second, one just has to take complex conjugation. This proves the equality.

Denote \( Y \) the image by \( X \) of the Albanese map and by \( U \) its dense set of smooth points. Since \( U \) is a complex submanifold of \( \text{Alb}(X) \), \( \int_U \omega^a(X) \) is well-defined and not 0. As in the proof of 3.5, this shows that \( \text{alb}^*\omega^a(X) \) is non-zero in \( H^a(X, a(X)) \).

Conversely since \( \text{alb}^* \) is an isomorphism on holomorphic 1-forms, a wedge \( \alpha_1 \wedge \cdots \wedge \alpha_{k+1} \) in \( H^{k+1,0} \) is of the form \( \text{alb}^*(\beta_1 \wedge \cdots \wedge \beta_{k+1}) \) with \( \beta_i \in H^{1,0}(\text{Alb}(X)) \). Since \( a(X) = k \) one has

\[
(\beta_1 \wedge \cdots \beta_{k+1})|_{\text{alb}^*X} = 0
\]

which implies that \( \alpha_1 \wedge \cdots \wedge \alpha_{k+1} = 0 \) too. This concludes the proof.
Part II

Kähler groups
Chapter 4

Motivations

Given a finitely presentable group, it is natural to look for a geometric interpretation in order to understand it better; a book of reference for this theory is [Ser80]. One of the simplest application of this principle is to consider the free group on \( k \) generators as the fundamental group of a wedge of \( k \) circles and then use the theory of covering spaces to prove the purely algebraic Schreier’s theorem:

**Theorem 4.1** (see [Pau09]). *Every subgroup of a free group is a free group.*

The general question is the following: Can every finitely presentable group be seen as the fundamental group of a nice space? The answer of this question will obviously depend on what the term nice means. We can also go in the other direction: a certain class of spaces being given, there may be interesting restrictions on a group in order to be the fundamental group of a space in this class.

As a corollary of the discussion in the chapter on algebraic topology, we get a first result.

**Proposition 4.2.** *Every (finitely presentable) group is the fundamental group of a (finite) CW-complex of dimension 2.*

4.1 Fundamental groups of smooth manifolds

The problem with the above construction is that a CW-complex cannot generally be given a structure of smooth manifold since we add cells of different dimensions. One can however obtain a similar statement by using a tool in differential topology, introduced by Milnor in [Mil61] and called *surgery*.

**Theorem 4.3.** *Every finitely presentable group is the fundamental group of a smooth closed orientable \( n \)-manifold for every \( n \geq 4 \).*

*Sketch of the proof.* Let \( G \) be a finitely presentable group with \( k \) generators and \( l \) relations. Let \( X \) be the connected sum of \( k \) copies of \( S^1 \times S^{n-1} \); this is a smooth closed orientable \( n \)-manifold with fundamental group the free group with \( k \) generators. The relations of \( G \) correspond to continuous maps from \( S^1 \) to \( X \). We can rearrange these maps in order that there images are embedded disjoint circles (here we use that \( n \geq 3 \), see [Hir76]).

Now one has to kill these circles in order to obtain the relations of \( G \). Consider disjoint tubular neighbourhoods \( S^1 \times D^{n-1} \) of the embedded circles. Notice that these neighbourhoods have the same boundary as \( D^2 \times S^{n-2} \), namely \( S^1 \times S^{n-2} \). For an embedded circle \( \iota : S^1 \to X \),
one can consider a new smooth orientable closed $n$-manifold $Z$ given by

$$Z := \left[ X - (S^1 \times D^{n-1}) \right] \cup_f D^2 \times S^{n-2},$$

where $f$ is the inclusion of the boundary $S^1 \times S^{n-2}$ in $X - (S^1 \times D^{n-1})$.

One can see that the fundamental group of $Z$ is the fundamental group of $X$ quotiented by the normal subgroup generated by $\iota_*(\pi_1(X))$. Indeed let $U$ be $Z - (\{0\} \times S^{n-2})$ and let $V$ be an open neighbourhood of $D^2 \times S^{n-2}$ in $X$ which retracts to $D^2 \times S^{n-2}$. Then $U$ is homotopic to $X - (S^1 \times D^{n-1})$ which itself is homotopic to $X - S^1$ and $V$ is simply connected - here we use $n \geq 4$. Since $X$ has dimension at least 4, $\pi_1(X - S^1) = \pi_1(X)$. Moreover the intersection of $U$ and $V$ retracts to $S^1 \times D^{n-1}$ whose fundamental group corresponds to $\iota_*(\pi_1(X))$. So the Van Kampen theorem proves the claim.

Doing this surgery for every embedded circle, we obtain a smooth closed orientable $n$-manifold, whose fundamental group is $G$. \hfill \Box

We can then ask which groups appear as fundamental groups of smooth closed orientable $n$-manifolds, for $n \leq 3$. For $n = 1$, we just get $\mathbb{Z}$ and for $n = 2$ the classification of orientable surfaces gives a list of non-isomorphic groups indexed by the genus of the surface. If $C_g$ is a surface of genus $g$, we have a presentation

$$\pi_1(C_g) = \langle a_1, \ldots, a_g, b_1, \ldots, a_g | [a_1, b_1] \ldots [a_g, b_g] \rangle.$$

For $n = 3$, things are more difficult.

### 4.2 Some cultural results

Apart the case of 3-manifolds, the question is solved in the class of smooth orientable closed manifolds. Now we can consider other classes, namely

- almost complex manifolds (see [Huy04] for definitions and basic properties);
- symplectic manifolds;
- complex manifolds.

We just quote the results; sketches of the proofs are in [ABC+96].

**Theorem 4.4.** Every finitely presentable group is the fundamental group of

- a closed almost-complex 4-manifold (Kotschick, [ABC+96] and [Kot92]);
- a closed symplectic 4-manifold (Gompf, [Gom95]);
- a complex 3-manifold, which can be taken symplectic (Taubes, [Tau92] and Gompf).

**Remark 4.1.**

- Making products with $S^2 = CP^1$ which is simply connected we get the same result for even real dimension $n \geq 4$ in the first two parts of the theorem and for complex dimension $n \geq 3$ in the third part.
• In the third part, the complex and symplectic structures are not compatible in general. Notice that the compatibility means that $X$ carries a Kähler structure.

• With these classes of manifolds, it only remains to see which groups happen to be fundamental groups of complex surfaces. We give some results in the next paragraph.

4.3 Smooth projective varieties versus Kähler manifolds

As we are mostly concerned with fundamental groups of compact Kähler manifolds, we make the following definition:

Definition 4.1. A Kähler group is a finitely presentable group which is the fundamental group of a compact Kähler manifold.

We give some results of existence and non-existence in the next chapter but we can already give simple results and examples.

Example 4.1.

• A Kähler group $G$ has even first Betti number (in characteristic 0). Indeed, this number is also the first Betti number of any compact Kähler manifold having $G$ as fundamental group. Hence it is smooth by Hodge theory.

• The abelian group $\mathbb{Z}^n$ is Kähler if and only if $n$ is even. If $n$ is even, then a complex torus $\mathbb{C}^n/\Lambda$ is Kähler and has fundamental group $\mathbb{Z}^n$. Conversely, the first Betti number of $\mathbb{Z}^n$ is the first Betti number of its classifying space $(S^1)^n$ which is $n$. Hence, $\mathbb{Z}^n$ cannot be Kähler for $n$ odd.

• Let $C_g$ be any compact smooth complex curve of genus $g$. There always exists a Hermitian metric on a complex manifold and this metric is automatically Kähler for a complex curve, since there are no forms in degree 3. Hence, we obtain a new family of Kähler groups.

• A product of two Kähler groups is a Kähler group since the product of two compact Kähler manifolds is a compact Kähler manifold for the product metric.

• A subgroup $H$ of finite index of a Kähler group $G$ is a Kähler group. Indeed, if $M$ is a compact Kähler manifold such that $\pi_1(M) = G$ then there is a covering $M'$ of $M$ with fundamental group $H$. Since the covering is finite, $M'$ is compact and a Kähler metric on $M$ lifts to a Kähler metric on $M'$.

Among all compact Kähler manifolds one can consider only those which can be embedded in some complex projective space. We call them smooth projective varieties. It is known that there are compact Kähler manifolds which are not projective and the Kodaira embedding theorem (see [Huy04]) gives a useful criterion to determine whether or not a Kähler manifold is projective. For instance, a complex torus of dimension $n \geq 2$ is generically not projective.

At the level of the fundamental groups, this is still an open problem.

Problem 4.5. Is every Kähler group the fundamental group of a smooth projective variety?

Another open problem discussed in [ABC+96] is the following.

Problem 4.6. Does every compact Kähler manifold deform to a smooth projective variety?
We will not go through the details but these deformations preserve the homotopy type of the manifold. In particular, a positive answer to problem 4.6 would imply a positive answer to problem 4.5. The Kodaira classification of surfaces (see [BHPV04]) implies that both problems have positive answer in complex dimension 2. Voisin recently proved that problem 4.6 has a negative answer in dimension $\geq 4$.

**Theorem 4.7** (see [Voi04]). *In any dimension $\geq 4$, there exist compact Kähler manifolds which do not have the homotopy type of a complex projective manifold.*

Hence, problem 4.6 remains open only in complex dimension 3 but nothing more seems to be known for problem 4.5. If the answer to the problem on fundamental groups were positive, then the problem of characterising Kähler groups would be a little easier because of the Lefschetz Hyperplane Theorem, which is proven in the next chapter.

**Theorem 4.8.** *Let $X \subset \mathbb{C}P^m$ be a smooth projective variety of complex dimension $m$ and $Y = X \cap H$ a generic hyperplane section. Then the natural morphism $\pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i \leq m - 2$ and is onto for $i = m - 1$.***

For a generic hyperplane, the section is still smooth. Beginning with $X$ a smooth projective variety of dimension $m$ one can thus obtain another smooth projective variety $Y$ of dimension $m - 1$ if $m \geq 3$. By iterating, this proves that every fundamental group of a smooth projective variety of dimension $\geq 2$ is already the fundamental group of one of complex dimension 2.

Hence we see that, if problem 4.5 has a positive answer, then the whole problem of classification of Kähler groups will be reduced to the classification of fundamental groups of smooth projective complex surfaces.
Chapter 5

Constructions of Kähler groups

In this chapter, we give some non-trivial constructions of Kähler groups. The idea of all these constructions is the following: first we construct a not-necessarily smooth projective variety whose fundamental group is interesting and then we cut the variety by a generic linear subspace in order to obtain a smooth compact projective variety. This is motivated by the Lefschetz Hyperplane Theorem which shows that, under certain hypotheses of smoothness, the fundamental group of the variety is stable under this cut.

5.1 Lefschetz Hyperplane Theorem

First we recall some definitions and results of Morse Theory needed in the proof of the theorem. Let $M^n$ be a compact manifold and let $f$ be a smooth real-valued function on $M$. A point $x \in M$ is a critical point of $f$ if the differential of $f$ at the point $x$ is 0. In coordinates $(x_1, \ldots, x_n)$ centered in $x$, this is equivalent to $\frac{\partial f}{\partial x_1}(0) = \cdots = \frac{\partial f}{\partial x_n}(0) = 0$. In these coordinates, the Hessian of $f$ at the critical point $x$ is the matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(0))_{i,j}$. We say that $x$ is a non-degenerate critical point if its Hessian is non-degenerate as a real bilinear form. In this case the index of $x$ is the index of its Hessian, the number of minus signs occurring in its signature. One checks that these notions of non-degeneracy and index do not depend on the coordinates at a critical point.

A Morse function on $M$ is a smooth real-valued function $f$ which has only non-degenerate critical points. Such a function always exists and can be taken as $C^2$-close as one wants from a given smooth function $g$ (see [Mat01], pages 47–56). Once such a function $f$ is fixed, we denote by $M_a$ the sublevel set $\{x \in M \mid f(x) \leq a\}$ where $a$ is an arbitrary real number. The idea of Morse theory is to study the deformation of the shape of $M_a$ when $a$ evolves along the whole real line. Since $M$ is compact, $f$ has a minimum $c_0$ and a maximum $c_1$, and we have that $M_{c_0-\epsilon} = \emptyset$ and $M_{c_1+\epsilon} = M$ for every $\epsilon > 0$. As the critical points of $f$ are non-degenerate they are isolated in a compact set; hence they are actually finite in number. The two main results are the following (see [Mil63])

**Theorem 5.1.** If $f^{-1}([a, b])$ contains no critical points then $M_a$ is diffeomorphic to $M_b$. Let $p$ be a non-degenerate critical point of $f$ with index $m$ and set $c = f(p)$. If $f^{-1}([c-\epsilon, c+\epsilon])$ contains no critical point of $f$ other than $p$ then $M_{c+\epsilon}$ has the homotopy type of $M_{c-\epsilon}$ with a $m$-cell attached.

With some technical work, we can always perturb a Morse function $f$ to a Morse function $g$ such that

- The functions $f$ and $g$ have the same number of critical points and, for every index, the number of critical points with this index coincide for $f$ and $g$.
• For critical points $p_i$ and $p_j$ of $g$, index$(p_i) <$ index$(p_j)$ implies $g(p_i) < g(p_j)$.

• Also, index$(p_i) =$ index$(p_j)$ implies $g(p_i) = g(p_j)$.

With such a Morse function $g$, we can attach simultaneously all the cells corresponding to the same critical value. Collecting these results, we obtain that every Morse function $g$ as above, defined on $M$, induces a structure of CW-complex on $M$. Moreover there is a one-to-one correspondence between the $m$-cells and the critical points of index $m$ for every integer $m \in \mathbb{N}$.

Following the paper of Raoul Bott [Bot59] we give a little generalisation of this result.

**Definition 5.1** (Non-degenerate manifolds). Let $\phi$ be a smooth real-valued function on a compact manifold $M$. A connected compact submanifold $V$ of $M$ is a non-degenerate critical manifold of $\phi$ if

- $d\phi = 0$ on $V$;
- for every $v \in V$, the nullspace of the Hessian of $\phi$ at $v$ is exactly the tangent space of $V$ at $v$. (By the first condition, the nullspace of the Hessian always contains the tangent space of $V$)

**Theorem 5.2.** Let $\phi$ be a smooth function on $M$ such that the critical points of $\phi$ consist of

1. for the minimum of $\phi$: a submanifold $M_*$ of $M$ whose connected components are non-degenerate critical manifolds of $\phi$;
2. for the other critical values of $\phi$: isolated critical points (which is the same as non-degenerate when considered as 0-submanifolds).

Then, up to homotopy, $M$ is obtained by attaching cells to $M_*$ and, for every integer $m$, the $m$-cells are in one-to-one correspondance with the critical points in $M - M_*$ of index $m$.

**Remark 5.1.** In order to prove this generalisation we only need to show that $M_*$ is an isolated manifold in the sense that it has a neighbourhood which does not contain any other critical point. This is a local statement since $M^*$ is compact and one can apply the Morse-Bott lemma which is a parametrized Morse Lemma; see [BH04].

We recall the notion of positivity for holomorphic line bundles (see [H09]). The notion of smooth section is also needed.

**Definition 5.2.** Let $(L, h)$ be a hermitian holomorphic line bundle over a complex manifold $X$. If $t$ is a local frame of $L$, then the weight $\phi$ of $(L, h)$ (relative to $t$) is defined by $\phi(z) = -\log(h(t(z), t(z)))$.

**Lemma 5.3.** With these notations, the curvature form $\Theta_L$ of the Chern connection of $(L, h)$ is given by

$$\Theta_L(z) = \partial \bar{\partial} \phi(z),$$

where $\phi$ is the weight of $(L, h)$ relative to any local frame $t$.

In particular, $\Theta_L$ defines a Hermitian product on $TX$ if and only if the matrix $(\frac{\partial \phi}{\partial z_i \partial \bar{z}_j})_{i,j}$ is positive definite. We say that $L$ is positive if there exists a metric $h$ on $L$ for which this is true.

**Definition 5.3.** With the same notations, a section $s$ of $L$ over $X$ is smooth if for every $x$ in the null-set of $s$, there exist local holomorphic coordinates $(z_1, \ldots, z_n)$ centered at $x$ and a section $s^*$ non-vanishing at $x$, such that $s = z_1 s^*$ around $x$. 
5.1. LEFSCHETZ HYPERPLANE THEOREM

With these recalls, we can prove a generalized version of the Lefschetz Hyperplane Theorem.

Theorem 5.4. Let $L$ be a positive line bundle over a compact complex manifold $X$ of complex dimension $n$, $s$ a holomorphic non-singular section of $L$ and let $S$ denote the null set of $s$. Then up to homotopy $X$ is obtained from $S$ by attaching cells of dimension $\geq n$.

Proof. By assumption there exists a positive hermitian metric $h$ on $L$. We have to prove three points.

Each component of $S$ is a non-degenerate critical manifold of the function $h(s,s)$.

Proof. Let $p$ be in $S$. Because $s$ is a non-singular section we have that $s(z) = z_1 s_1(z)$ in some holomorphic coordinates $(z_1, \cdots, z_n)$ centered in $p$, where $s_1$ is a local section of $E$ non vanishing at $p$. So $S$ is locally given near $p$ by the equation $z_1 = 0$ and $S$ is a smooth manifold. Moreover $h(s(z),s(z)) = |z_1|^2 h(s_1(z), s_1(z))$ so the differential of $h(s,s)$ vanishes at $p$ which is 0 in the coordinates. In order to show that the component of $S$ in which $p$ lies is non-degenerate, let $(x_1, x_2)$ be real coordinates such that $z_1 = x_1 + ix_2$. If we set $a(z) = h(s_1(z), s_1(z))$, we have

\[
\frac{\partial^2 h(s(z),s(z))}{\partial x_i \partial x_j} = 2x_i a(z) + |z_1|^2 \frac{\partial a}{\partial x_i}. \quad \text{So } \frac{\partial^2 h(s(z),s(z))}{\partial x_i \partial x_j} |_{z=p} = 2a(p)\delta_{ij}. \quad \text{Since } a(p) > 0, \text{ this proves the non-degeneracy.}
\]

Let $p$ be a critical point of $h(s,s)$ in $X - S$. Then its index is no less than $\text{dim}_{\mathbb{C}}(X) = n$.

Proof. On $X - S$, $s$ is a frame of $L$ and we can consider $f = - \log h(s,s)$. By the facts recalled at the beginning of the proof $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j}$ is positive definite. Consider $H$ the Hessian of $f$ at $p$ in coordinates $x_i, y_j$ where $z_j = x_j + iy_j$. Then we can extend $H$ to an hermitian form $H'$ in the complexified tangent space. The index of $H$ as a bilinear form is the index of $H'$ as an hermitian form. Since $H'(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = (\frac{\partial^2 f}{\partial z_i \partial z_j})$, $H'$ is positive definite in a complex subspace of complex dimension $n$; hence the index of $H$ is at most $n$. Finally, by a direct computation:

\[
\frac{\partial h(s,s)}{\partial x_i} = -e^{-f} \frac{\partial f}{\partial x_i}; \quad \text{and} \quad \frac{\partial^2 h(s(p),s(p))}{\partial x_i \partial x_j} = e^{-f} \left( \frac{\partial f(p)}{\partial x_i} \frac{\partial f(p)}{\partial x_j} - \frac{\partial^2 f(p)}{\partial x_i \partial x_j} \right). \tag{5.1, 5.2}
\]

The first part in the right hand side of (5.2) is zero because of (5.1) - remember that $p$ is a critical point of $h(s,s)$. So the index of $h(s,s)$ is the index of $-f$. This proves the point. \[\square\]

The critical points of $h(s,s)$ outside $S$ are non-degenerate.

In fact this could be false so we need to perturb a little the function $h(s,s)$ without losing the two preceding points. We need two lemmas for this.

Lemma 5.5. Fix a Riemannian metric and a connection on $M$ so that one can speak of the norm of the derivatives of any order. Let $F$ be a smooth function on $M$. Then there exists a smooth function $\eta$ ($C^2, \epsilon$)-small such that $F + \eta$ is a Morse function. Here ($C^2, \epsilon$)-small means that the function is bounded in absolute value by $\epsilon$ and its first and second derivatives too where these derivatives are computed with the connection.

We will not prove this approximation lemma. It is actually used to show the existence of Morse functions; the principal ingredients are Sard’s lemma and arguments with partitions of unity. We refer again to [Mat01], pages 46 – 57.

Now let $A$ and $B$ be open neighbourhoods of $S$ such that

- $A \subset B;$
• $h(s, s)$ has no critical points in $B$ other than $S$.

This is possible because $S$ is non-degenerate (see the remark 5.1). Moreover take $g$ a cut-off function which is 0 on $A$ and 1 on $X - B$.

**Lemma 5.6.** There exists $\epsilon_1$ such that if $\eta$ is a $(C^2, \epsilon_1)$-small function and if we set $\phi = h(s, s) + g\eta$ then

- $\phi$ is positive on $X - A$;
- $\phi$ has no critical points on the closure of $B - A$;
- In local holomorphic coordinates $(\frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j})_{i,j}$ is positive definite on $X - A$.

**Proof.** First note that if $\eta$ is $C^2$-small then so is $g\eta$. More precisely there exists a constant $M$ depending on $g$ and its first and second derivatives such that if $\eta$ is $(C^2, \epsilon)$-small then $g\eta$ is $(C^2, M\epsilon)$-small as one can see by computing the derivatives.

Then we see that the three conditions are locally open in the $C^2$-topology; this means that if one of these conditions is satisfied for some function $f$ at a point $p$ then for a sufficiently $C^2$-small function $\eta$ the condition will also be satisfied at $p$ for $f + \eta$. But then the condition is also satisfied in a neighbourhood of $p$ because $f + \eta$ and its derivatives are continuous.

Finally these conditions are in fact open in the $C^2$-topology because $X - A$ and $B - A$ is compact. As the three conditions are satisfied for $h(s, s)$, this proves the lemma. \qed

We can finish the proof of the theorem. By the approximation lemma let $\eta$ be $(C^2, \epsilon_1)$-small (with $\epsilon_1$ as in the previous lemma) such that $h(s, s) + \eta$ is a Morse function. Then set $\phi = h(s, s) + g\eta$. By the first condition $\phi$ has the same zero set as $h(s, s)$, namely $S$. Moreover since $\phi$ and $h(s, s)$ coincide on $A$, the components of $S$ are non-degenerate manifolds for $\phi$ too. By the first and third conditions, it is still true that the critical points of $\phi$ outside $S$ have index no less than $n$; these were indeed the only ingredients for the proof of this point. Finally, $\phi$ is non-degenerate at its critical points outside $S$. Indeed by the second condition these critical points are outside $B$ and in $X - B$, $\phi = h(s, s) + g\eta$ which only have non-degenerate critical points by construction. This concludes the proof of the theorem. \qed

As a corollary we obtain the Lefschetz Hyperplane Theorem.

**Theorem 5.7 (Lefschetz Hyperplane Theorem).** Let $X \subset \mathbb{C}P^n$ be a smooth projective variety of complex dimension $m$ and let $Y = X \cap H$ be a generic hyperplane section. Then the natural morphism $\pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i \leq m - 2$ and is onto for $i = m - 1$.

**Proof.** Consider the positive line bundle $E = O(1)$ over $\mathbb{C}P^n$ whose global sections parametrize the hyperplanes in $\mathbb{C}P^n$. By Bertini’s theorem (see 1.13), the generic section of $X$ with a hyperplane is a smooth projective manifold and the corresponding section $s$ is smooth over $X$.

We apply the theorem 5.4 with the restriction of $E$ over $X$ and denote by $Y$ the intersection of $H$ and $X$ for such a generic section. Then we have that, up to homotopy, $X$ is obtained by attaching cells of dimension at least $m$ to $Y$. By the arguments used in corollary 2.4 this implies that the natural morphism $\pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i \leq m - 2$ and is onto for $i = m - 1$. \qed

We give a generalization of Lefschetz Hyperplane Theorem that we will need in the third and fourth sections of the chapter.

**Theorem 5.8 (Goresky-Mac Pherson [GMP88]).** Let $V$ be a smooth projective variety and $f : V \to \mathbb{C}P^n$ a holomorphic map with finite fibers. If $L \subset \mathbb{C}P^n$ is a generic linear subspace, then the natural morphism $\pi_i(f^{-1}(L)) \to \pi_i(V)$ is an isomorphism for $i \leq m - 1$ and is onto for $i = m$, where $m = \dim_{\mathbb{C}} f^{-1}(L)$. 
5.2. **Serre theorem**

A first application of the Lefschetz Hyperplane Theorem is the following theorem due to Serre.

**Theorem 5.9.** *Every finite group is Kähler.*

We first give an informal idea of the proof. Because every finite group is a subgroup of some symmetric group, it is sufficient to prove the assertion for symmetric groups $S_m$. Indeed, if $S_m$ is a covering map of structural group $S_m$ outside some singularity locus corresponding to the ramification points. Hence we obtain $S_m$ as the fundamental group of some open set in a projective variety. The technical part of the proof - where we use at the end the Lefschetz Hyperplane Theorem - consists of the construction of a compact Kähler manifold from this open set.

**Proof.** As explained above we can assume that the finite group $\Gamma$ is $S_m$, for some $m > 1$. Let $s$ be a positive integer and consider the product $X = \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^s$ with $m$ factors and the natural action of $S_m$ given by permutation of factors. We introduce $s + 1$ new indeterminates $t_0, \ldots, t_s$ and write $t$ for $(t_0, \ldots, t_s)$ and $x$ for a point $(x_1, \ldots, x_m) = (x_0, \ldots, x_{s+1}, \ldots, x_0, \ldots, x_m)$ in $X$. We define

$$F(x, t) = \prod_{j=1}^{m} L_j(x_j, t),$$

where $L_j(x_j, t) = \sum_{i=0}^{s} x_{ij}t_i.$

We denote the monomials of degree $m$ in the variables $t_0, \ldots, t_s$ by $T^\alpha$ and their number by $M + 1 = \binom{m+s}{m}$. We can then write

$$F(x, t) = \sum_{\alpha=0}^{M} F_\alpha(x)T^\alpha.$$  

This defines a map from $X$ to $\mathbb{C}P^m$ by

$$\phi(x) = [F_0(x) : \cdots : F_\alpha(x) : \cdots : F_M(x)],$$

where $\alpha$ runs over the integers from 0 to $M$. This mapping is well defined. Indeed each $F_\alpha(x)$ is linear in each system $x_{0j} \ldots x_{sj}$ so that the image in $\mathbb{C}P^m$ does not depend on representatives in each factor $\mathbb{C}P^s$ of $X$. Moreover if $F_\alpha(x)$ were 0 for every $\alpha$ and some point $x$ in $X$ then $F(x, t) = 0$ and necessarily one $L_j(x_j, t)$ would be 0 which is impossible.

Notice that each fiber of this map corestricted to its image $X' = \phi(X)$ consists of the orbit of a point $x \in X$ by the action of $S_m$. Indeed (5.3) shows that the mapping is constant on each orbit and conversely if $\phi(x) = \phi(y)$ then $F(x, t) = \lambda F(y, t)$ as polynomials in $t$. The uniqueness of factorisation in $\mathbb{C}[t]$ implies that $L_j(x_j, t) = c_jL_j(y_{ij}, t)$ where $c_j$ is a constant and $j \mapsto i_j$ is a permutation of $\{1, \ldots, m\}$.

Denote by $W$ the points in $X$ such that $x_i \neq x_j$ for $0 \leq i \neq j \leq m$ and by $W'$ its image by $\phi$. Every fiber corresponding to a point in $W'$ has cardinal $m!$ and the restriction $\phi : W \rightarrow W'$ is a covering map of structural group $S_m$. Indeed if $x$ is in $W$ we can choose disjoint open sets $U_j$ in $\mathbb{C}P^s$ such that $x_j \in U_j$. Hence $(\prod_j U_j(j))_{j \in S_m}$ are disjoint open sets and each product contains exactly one point in the orbit of $X$. As they are mapped to some neighbourhood $V$ of $\phi(x)$ this gives a local trivialisation of $\phi$ for a covering map with structural group $S_m$. 


Now we need to cut $W'$ with hyperplanes in $\mathbb{CP}^M$ in order to obtain a compact projective manifold with the same properties. If we denote by $\Delta$ and $\Delta'$ respectively the complementary in $X$ of $W$ and its image by $\phi$ we have that both $X$ and $\Delta$ are projective varieties. This is true for $X$ by the Segre embedding $\iota$ and for $\Delta$ since it is given by some equations $x_i = x_j$ which translate into equalities of monomials involving the coordinates of $x_i$ and $x_j$ in the projective variety $\iota(X)$. Hence both $X'$ and $\Delta'$ are closed subvarieties of $\mathbb{CP}^m$ since they are images of closed projective subvarieties by a regular mapping (see [Sha94]). Moreover $\phi$ preserves the dimensions of the varieties since its fibres are all 0-dimensional. In particular:

$$\dim(\Delta') = \dim(\Delta) = (m - 1)s. \quad (5.4)$$

By corollary (1.14), if $d > (m - 1)s$, the linear subspace $L'$ of codimension $d$ in $\mathbb{CP}^m$ is such that $L'$ does not intersect $\Delta'$ and its intersection $Y'$ with $X'$ is smooth. We just have to prove that the inverse image of $Y$ by $\phi$ is connected and simply connected. Note that by definition of $\iota$ and $\phi$ there exists a linear projection $p$ from $\iota(X)$ to $X'$ and such that

$$p \circ \iota = \phi.$$ 

This implies that the inverse image $Y$ of $Y' = L' \cap X'$ is given by the intersection $X \cap L$ for some linear subspace $L$ of $\mathbb{CP}^N$ of codimension $d$. Moreover $X$ is smooth as it is the inverse image of $Y'$ by a submersion. The Lefschetz Hyperplane Theorem then shows that $X$ is connected and simply connected if $\dim(Y) = \dim(X) - d = m.s - d \geq 2$. Since the only other restriction is $d > (m - 1)s$, one can take $s \geq 3$ and $d = ms - 2$ and both conditions are fulfilled.

\[\square\]

5.3 Construction of a Kähler group with odd cohomological dimension

We just give the general idea of the construction of a Kähler group, which uses some results about lattices. It was given in [To90] and is also discussed in [ABC+96], p.111–112.

Let $\Gamma$ be a torsion-free lattice of finite index of $SU(n, 1, \mathbb{Z}[i])$, the group of $(n + 1) \times (n + 1)$ matrices with coefficients in $\mathbb{Z}[i]$ and which preserve the bilinear form $h$ defined by $h(z) = |z_0|^2 - |z_1|^2 - \cdots - |z_n|^2$. Denoting by $B^n$ the open unit ball in $\mathbb{C}^n$ and embedding it in $\mathbb{CP}^n$, one sees easily that $\Gamma$ acts on $B^n$.

One shows that the smooth quotient $V = \Gamma \backslash B^n$ has a projective compactification: there exists $\bar{V}$ a projective variety such that $V$ is open in $\bar{V}$ and $\bar{V} - V$ is finite. If we cut $V$ by a generic hyperplane $H$, we avoid this finite set of points and obtain a smooth projective variety $V \cap H$. Moreover, by theorem 5.8, (taking the inclusion for $f$), we get isomorphisms $\pi_1(V \cap H) \cong \pi_1(V)$ for $i \leq n - 2$. Since $\pi_1(V) = \Gamma$, we get

\textbf{Corollary 5.10} (Corollary 8.4 in [ABC+96]). \textit{If} $n \geq 3$, \textit{then} $\Gamma$ \textit{is a Kähler group.}

\textbf{Remark 5.2}. $V$ is a quotient of $B^n$, hence it is an Eilenberg-Mac Lane space $K(\Gamma, 1)$. Since $V$ is a non-compact smooth manifold of real dimension $2n$, it has no cohomology in degree $\geq 2n$. We say that $V$ has \textit{cohomological dimension} at most $2n - 1$. For some groups $\Gamma$, the cohomological dimension will be exactly $2n - 1$. It is not known whether there exists or not a Kähler group with cohomological dimension 3 (see 8.5).

5.4 A general construction

We finally give a very general construction of Kähler groups, using Goresky-Mac-Pherson theorem [GMP88].
Let $L_1$ and $L_2$ be two $n$-dimensional linear subspaces in general position in $\mathbb{C}P^{2n+1}$ and let $U$ be the complementary of their union. Every point $x$ in $U$ lies in a unique line $L_x$ which intersects both $L_1$ and $L_2$. Hence there is a natural map $U \rightarrow L_1 \times L_2$ given by $x \mapsto (L_1 \cap L_x, L_2 \cap L_x)$. The fiber of this map over any $(z_1, z_2)$ in $L_1 \times L_2$ is $\mathbb{C}P^1$ minus two points, that is $\mathbb{C}^\ast$. Moreover one checks that this map is a fibration.

Consider $A$ et $B$ two projective $n$-dimensional manifolds and $\alpha : A \rightarrow L_1$ and $\beta : B \rightarrow L_2$ two finite maps. Denote by $V$ the pullback of the fibration $U \rightarrow L_1 \times L_2$ by the map $(\alpha, \beta)$ so that we have a commutative diagram

$$
\begin{array}{ccc}
V & \rightarrow & U \\
\downarrow & & \downarrow \\
A \times B & \rightarrow & L_1 \times L_2
\end{array}
$$

Since the bottom map is finite, the top map is also finite. Hence we get a finite map $V \rightarrow \mathbb{C}P^n$. A generic $n$-dimensional linear subspace $\Lambda$ in $\mathbb{C}P^{2n+1}$ lies in $U$ and is transversal to $f$. Hence, $M = f^{-1}(\Lambda)$ is a smooth projective variety and, by theorem [GMP88], $\pi_1(M) \cong \pi_1(V)$ if $n \geq 2$. This shows that $\pi_1(V)$ is Kähler.

Since by definition, we have a fibration $V \rightarrow A \times B$ of fiber $\mathbb{C}^\ast$, we get an exact sequence

$$
\pi_2(A \times B) \rightarrow \mathbb{Z} \rightarrow \pi_1(V) \rightarrow \pi_1(A \times B) \rightarrow 1.
$$

With this construction, one can notably construct Kähler groups which are residually finite, that is Kähler groups $\Gamma$ such that

$$
\bigcap_{\Delta \subset \Gamma, \text{finite index}} \Delta \neq 1.
$$

We refer to the fifth part of chapter 8 in [ABC+96].
Chapter 6

Restrictions on Kähler groups

Several restrictions are known for Kähler groups. They are obtained by different techniques: \(L^2\)-methods, rational homotopy theory, linear representations... In the first section of this short chapter, we give without proof some of these restrictions. In the second section, we concentrate on the notion of formality and obtain another one. The book of reference is [ABC+96].

6.1 Panorama of the results

We first recall a useful notion in group theory.

**Definition 6.1.** Let \(G\) be a group and \(P\) a property in group theory. We say that \(G\) has *virtually* \(P\) if \(G\) has a finite index subgroup satisfying \(P\).

Since a finite index subgroup of a Kähler group is Kähler, as soon as we know that a Kähler group cannot satisfy some property \(P\) then it cannot be virtually \(P\) neither. For instance a group which has virtually an odd first Betti number is not Kähler.

**Example 6.1.** \(\mathbb{Z} \ast \mathbb{Z}\) contains \(\mathbb{Z}\) as a subgroup of order 2, hence it is not Kähler.

A subtler remark is the following. By proposition 3.4, we have a natural non-degenerate bilinear pairing on \(H^1(X, \mathbb{C})\). Since \(H^1(X, \mathbb{C})\) and \(H^1(\pi_1(X), \mathbb{C})\) are isomorphic, this pairing can be seen on \(H^1(\pi_1(X), \mathbb{C})\). Moreover, since the natural map \(H^2(\pi_1(X), \mathbb{C}) \to H^2(X, \mathbb{C})\) is a monomorphism by theorem 2.8, we get

**Proposition 6.1.** Let \(G\) be a Kähler group. There exists a non-degenerate bilinear pairing on \(H^1(\pi_1(X), \mathbb{C})\) which factors through the cup-product:

\[
H^1(G, \mathbb{C}) \times H^1(G, \mathbb{C}) \xrightarrow{\cup} H^2(G, \mathbb{C}) \to \mathbb{C}.
\]

**Example 6.2.** In particular \(b_2(G, \mathbb{C})\) (or more precisely the image of the cup-product in \(H^2(G, \mathbb{C})\)) cannot be zero unless \(b_1(G, \mathbb{C})\) is zero.

The following results are deeper. We just give references for the proofs.

**Theorem 6.2** (Corollary 4.2 of [ABC+96], p. 47). A Kähler group is never a non-trivial free product.

**Theorem 6.3** (Theorem 6.22 of [ABC+96], p. 82). The fundamental group of a real hyperbolic manifold of dimension \(n \geq 3\) is never a Kähler group.

Both theorems will be used for the proof of theorem 8.6.

**Theorem 6.4.** Let \(G\) be a group which is both the fundamental group of a closed 3-manifold and a Kähler group. Then \(G\) is finite.
6.2 Formality of compact Kähler manifolds

In this section we show that the de Rham complex of a compact Kähler manifold satisfies the condition of *formality* and explain the consequences for the cohomology of a Kähler group. The interesting algebraic structure is the commutative differential graded algebra.

**Definition 6.2.** A (real) **commutative differential graded algebra** (CDGA) is a graded $\mathbb{R}$-algebra $A$ with a boundary operator $d: A \to A$ of degree 1 such that

- $y \cup x = (-1)^{|x||y|}x \cup y$;
- $d^2 = 0$ and $d(x \cup y) = dx \cup y + (-1)^{|x|}x \cup dy$,

where $|x|$ and $|y|$ are the degrees of $x$ and $y$.

**Example 6.3.** The de Rham complex of any smooth manifold $X$ is a CDGA. The (graded) cohomology algebra of any CDGA, is a CDGA with boundary operator $d = 0$.

**Definition 6.3.**

- A **morphism** of CDGAs is a morphism of algebras that respects the degree and the boundary operator.
- A **quasi-isomorphism** of CDGAs is a morphism of CDGAs that induces an isomorphism on cohomology.
- Two CDGAs $A$ and $B$ are **weakly equivalent** if there exists an integer $n$ and CDGAs $(C_1, \ldots, C_{n-1})$ such that for all $i$ in $\{0, \ldots, n-1\}$, there exists a quasi-isomorphism from $C_i$ to $C_{i+1}$ or from $C_{i+1}$ to $C_i$, with $C_0 = A$ and $C_n = B$.

**Definition 6.4.** A smooth manifold $X$ is **formal** if its de Rham complex $\mathcal{E}^*(X)$ and its real cohomology $H^*(X, \mathbb{R})$ (with boundary operator $0$) are weakly equivalent.

The main result is the following.

**Theorem 6.5.** *Compact Kähler manifolds are formal.*

**Proof.** The proof is more technical than complicated. It is well detailed in [ABC+96], page 33, and we refer to it. □

This theorem can be seen as a restriction on the cohomology of a Kähler manifold. We need another definition to be more concrete.

**Definition 6.5.** Let $A$ be a CDGA and $H$ its cohomology. Let $\alpha, \beta, \gamma$ be in $H$, of degrees $p, q$ and $r$ such that $\alpha \cup \beta = \beta \cup \gamma = 0$. Then choose $a, b, c$ in $A$ which are sent to $\alpha, \beta, \gamma$ and choose also $f$ and $g$ in $A$ such that $df = a \cup b$ and $dg = b \cup c$. The **Massey triple product** $\langle \alpha, \beta, \gamma \rangle$ is the class of $f \cup c + (-1)^{p-1}a \cup g$, which is well-defined in $H^{p+q+r-1}/(\alpha \cup H^{q+r-1} + \gamma \cup H^{p+q-1})$, as one easily checks.

**Proposition 6.6.** *On a formal space, all Massey triple products are zero.*

**Proof.** If $A$ and $B$ are linked by a quasi-isomorphism, one can compute the Massey triple products of $A$ in $B$ by naturality of the definition. Hence, one can compute the Massey products of a formal space directly in its cohomology algebra. Since the cohomology algebra is by definition endowed with the boundary operator $d = 0$, one can take $f = g = 0$ in the definition of the Massey products. This proves that all Massey products are zero. □
Corollary 6.7. Let $\Gamma$ be a Kähler group. Then all Massey triple products $\langle \alpha, \beta, \gamma \rangle$ of classes in $H^1(\Gamma, \mathbb{R})$ are zero.

Proof. Let $X$ be a compact Kähler manifold with fundamental group $\Gamma$. One can identify $H^1(X, \mathbb{R})$ and $H^1(\Gamma, \mathbb{R})$ via the classifying m of $X$ and see $H^2(\Gamma, \mathbb{R})$ as a subspace of $H^2(X, \mathbb{R})$. By the previous corollary, $\langle \alpha, \beta, \gamma \rangle$ is zero as a class of $H^2(X, \mathbb{R})/ (\alpha \cup H^1(X, \mathbb{R}) + \gamma \cup H^1(X, \mathbb{R})) = H^2(\Gamma, \mathbb{R})/ (\alpha \cup H^1(\Gamma, \mathbb{R}) + \gamma \cup H^1(\Gamma, \mathbb{R}))$, hence it is zero in $H^2(\Gamma, \mathbb{R})/ (\alpha \cup H^1(\Gamma, \mathbb{R}) + \gamma \cup H^1(\Gamma, \mathbb{R}))$. \hfill \Box

Example 6.4. Denote by $\mathcal{H}_3(\mathbb{Z})$ the Heisenberg group of matrices
\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\]
with integral coefficients. One shows (see [ABC+96], page 34) that the 1-minimal model of a topological space $X$ with fundamental group $\mathcal{H}_3(\mathbb{Z})$ is $M = \wedge(x, y, z)$ (the free graded commutative differential algebra generated by elements $x$, $y$ and $z$) with $x$, $y$ and $z$ of degree 1 and $dx = 0$, $dy = 0$, $dz = x \cup y$. This means that there exists a morphism of CDGAs from $M$ to the de Rham complex of $X$ inducing isomorphism in $H^0$ and $H^1$ and a monomorphism on $H^2$. In particular a Massey triple product of classes in $H^1(X, \mathbb{R})$ will be zero if and only if it is zero when computed in the cohomology of $M$.

The classes of $x$ and $y$ generate $H^1(M)$ and the cup-products $x \cup x$ and $x \cup y$ are zero in cohomology since $dz = x \cup y$. We can thus consider the Massey product $\langle \bar{x}, \bar{x}, \bar{y} \rangle$ which is the class of $0 \cup y + x \cup z = x \cup z$ in $H^2(M)/ (\bar{x} \cup H^1(M) + \bar{y} \cup H^1(M)) = H^2(M)$. Hence this Massey triple product is non-zero and the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ is not Kähler.

Remark 6.1. One can also remark that $H^1(\mathcal{H}_3(\mathbb{Z}), \mathbb{R})$ is 2-dimensional but the cup-product on $H^1$ is trivial. This gives a simpler proof that this group is not Kähler (see 6.2) but the previous proof shows more generally that $\mathcal{H}_3(\mathbb{Z})$ cannot be the fundamental group of any formal space.
Part III

The interaction between $\pi_1(X)$ and the geometry of $X$
Chapter 7

Fibration over complex curves

In order to study the geometrical properties of a complex manifold $X$, one can try to see $X$ as a fibration over a simpler manifold. In this chapter and the following, we will be interested in fibrations over complex curves. The main result of this chapter is the Siu-Beauville theorem, which asserts that the genera of the curves on which $X$ can fiber are determined only by its fundamental group $\pi_1(X)$.

7.1 Structure of a holomorphic map to a complex curve

Let $X$ be a compact complex manifold and let $C$ be a (smooth compact connected) complex curve. If $f$ is a non-constant holomorphic map from $X$ to $C$ then it is surjective. From now on, we thus assume that $f$ is surjective. The critical points of $f$ in $X$ form an analytic set, hence the proper mapping theorem implies that the critical values form an analytic set in $C$ since $X$ is compact. Moreover, by Sard’s lemma, the set of critical values has no interior, hence it has dimension 0 (see 1.8). Since $Y$ is compact, this set consists only in a finite number of points $\{a_1, \ldots, a_k\}$.

Denote by $C^*$ the open set $C - \{a_1, \ldots, a_k\}$ and by $X^*$ its inverse image by $f$. By definition, $f$ is submersive over $Y^*$. Let us recall the classical theorem of Ehresmann.

Theorem 7.1 (Ehresmann’s fibration theorem, [Huy04], p. 269). A smooth map $f$ between $M$ and $N$ smooth manifolds, which is a submersive surjection and a proper map, is a locally trivial fibration.

The restriction $\tilde{f}$ of $f$ from $X^*$ to $C^*$ satisfies the assumptions of the theorem, hence is a locally trivial fibration. In particular, two closed fibers of $f$ over points in $C^*$ are diffeomorphic. Since $C^*$ is connected, they are in fact all diffeomorphic.

Remark 7.1. Note that we say nothing about the complex structure on the fibers. In general, the fibers will not be biholomorphic.

The above discussion shows that we can speak about the diffeomorphism type of the generic fiber (i.e. not over a critical value). This allows us to define what we call a fibration in this context.

Definition 7.1. Let $X$ be a compact Kähler manifold and let $f$ be a holomorphic map from $X$ to a compact complex curve $C$. We say that $f$ is a fibration if it is surjective and if its generic fiber is connected. We say that $X$ fibers over $C$ if there exists such a fibration $f$ from $X$ to $C$.

Remark 7.2. If $f$ is a surjective holomorphic map from $X$ to a (possibly singular) compact curve $C$, then the Stein factorization of $f$ provides a fibration of $X$ over a possibly different smooth compact curve $D$. We will use this construction several times.
To finish the presentation of fibrations, we prove the following easy but useful lemma.

**Lemma 7.2.** If \( f : X \to C \) is a fibration, then \( f_* \) is surjective on \( \pi_1 \).

**Proof.** A loop \( \gamma : S^1 \to C \) is homotopic to a map \( \gamma' : S^1 \to C^* \) since we only have to avoid a finite number of points. Choose a point \( y_0 \) in the fiber of \( f \) over \( x_0 := \gamma'(0) \). Since \( \tilde{f} \) is a locally trivial fibration, we can locally lift the loop \( \gamma' \) in \( X \). By compactness, the whole loop \( \gamma' \) can be lifted to a path \( \tilde{\gamma} \) in \( X \). The point \( y_1 := \tilde{\gamma}(1) \) is in the same fiber of \( y_0 \). Since this fiber is pathwise connected (it is a connected manifold), one can join the two points \( y_0 \) and \( y_1 \) and reparametrize. This gives a path in \( X \) which is sent to a path homotopic to \( \gamma' \) in \( C \). \( \square \)

### 7.2 Castelnuovo-de Franchis and Catanese theorems

A powerful tool to construct fibrations over curves is the Castelnuovo-de Franchis theorem. We begin by a definition and a quite technical lemma.

**Definition 7.2.** Let \( X \) be a compact Kähler manifold.

- A subspace \( V \subset H^0(X, \Omega_X) \) is *isotropic* if the cup-product (for global holomorphic 1-forms) restricted to \( \Lambda^2 V \) is identically 0.

- A subspace \( U \subset H^1(X, \mathbb{R}) \) (resp. \( H^1(X, \mathbb{C}) \)) is *isotropic* if the cup-product (for cohomology) restricted to \( \Lambda^2 U \) is identically 0.

**Lemma 7.3.** If \( \omega_1 \) and \( \omega_2 \) are two linearly independant closed holomorphic 1-forms satisfying \( \omega_1 \wedge \omega_2 = 0 \) on a connected complex manifold \( M \), then the meromorphic function \( \frac{\omega_1}{\omega_2} \) extends to a holomorphic function \( M \to \mathbb{C}P^1 \).

**Proof.** See [NR97], p. 1353 – 1354. \( \square \)

**Remark 7.3.** The condition of closedness is automatically satisfied if \( M \) is a compact Kähler manifold.

**Theorem 7.4** (Castelnuovo-de Franchis). Let \( X \) be a compact Kähler manifold and \( V \subset H^0(X, \Omega_X) \) a \( g \)-dimensional isotropic subspace with \( g \geq 2 \). Then there exists a fibration \( f \) to a curve \( C \) such that \( V \subset f^*H^0(C, \Omega_C) \). Necessarily the genus of the curve is at least \( g \).

**Proof.** Let \( \omega_1, \ldots, \omega_g \) be a basis of \( V \). Let \( U_j \subset X \) be the open set where \( \omega_j \) is not zero and denote by \( U \) the union of the \( U_j \).

On \( U_j \), \( \omega_i \) and \( \omega_j \) are proportional since \( \omega_i \wedge \omega_j = 0 \). So we can define maps \( \phi_{ij} \) from \( U_j \) to \( \mathbb{C} \) such that \( \omega_i = \phi_{ij}\omega_j \). Since the \( \omega_i \) and \( \omega_j \) are holomorphic 1-forms, \( \phi_{ij} \) is holomorphic too.

We define a holomorphic map \( \phi_j \) by

\[
\phi_j : U_j \to \mathbb{C}P^{g-1} \\
p \mapsto [\phi_{1j}(p) : \ldots : \phi_{gj}(p)].
\]

On \( U_j \cap U_k \), \( \phi_j \) and \( \phi_k \) coincide since we have \( \omega_i = \phi_{ij}\omega_j = \phi_{ij}\phi_{jk}\omega_k \) which implies that \( \phi_{ik} = \phi_{ij}\phi_{jk} \). Hence we can define a holomorphic map \( \phi \) from \( U \) to \( \mathbb{C}P^{g-1} \) which is \( \phi_j \) on \( U_j \).

We claim that \( \phi \) extends to a holomorphic map from \( X \) to \( \mathbb{C}P^{g-1} \). By the above lemma, the quotient \( \frac{\omega_j}{\omega_j} \) can be considered as a holomorphic map from \( X \) to \( \mathbb{C}P^1 \) and equals \( \phi_{ij} \) on \( U_j \). Let \( j \) be in \( \{1, \ldots, g\} \). We consider the open set \( X_j = \{ p \in X \mid \forall i, \frac{\omega_i(p)}{\omega_j(p)} \neq \infty \} \) which contains \( U_j \). One can extend holomorphically \( \phi_j \) to \( X_j \). All extensions are compatible because
U is dense in X. If \( p \in X \), \( p \in X_k \) if \( k \) is a minimal element in \( \{1, \ldots, g\} \) for the strict order \( i < j \Leftrightarrow \omega_i(p)/\omega_j(p) = \infty \). Hence \( X = \bigcup_j X_j \) and we have the desired extension of \( \phi \).

For all \( p \) in \( U \) denote by \( F_p \subset T_p X \) the intersection of all \( \ker(\omega_i(p)) \). One has that \( F_p \) is an hyperplane of \( T_p X \) since the \( \omega_j \) are proportional to one another and not all zero for \( p \in U \). On \( U_j \), the distribution \( F \) is given by a single equation \( \omega_j = 0 \). Since \( \omega_j \) is holomorphic, it is closed by Hodge theory and so the distribution \( F \) (only defined on \( U \)) is integrable.

We show that \( \phi \) is constant on the connected components of the leaves of the foliation. This is equivalent to \( d\phi_j(\ker(\omega_j)) = 0 \) on \( U_j \). But since the \( \omega_i \) are all closed, one has

\[
0 = d\omega_i = d\phi_{ij} \wedge \omega_j + \phi_{ij} d\omega_j = d\phi_{ij} \wedge \omega_j.
\]

If \( X_p \) is in \( \ker(\omega_j(p)) \) but \( Y_p \) is not, \( (d\phi_{ij} \wedge \omega_j)(X_p, Y_p) = 0 \) implies that \( d\phi_{ij}(X_p)\omega_j(Y_p) = 0 \); hence \( d\phi_j(\ker(\omega_j)) = 0 \). This proves that \( \phi \) is constant on the connected components of the leaves of the foliation. Since they have dimension \( \dim(X) - 1 \), the image of \( \phi_U \) is of dimension at most 1 at points in the image \( \phi(U) \). By density of \( \phi(U) \) in \( \phi(X) \), the entire image is at most 1-dimensional. As \( g \geq 2 \), we have for instance that \( \phi_{12} \) is not constant on \( U_2 \), which implies that \( \phi \) is not constant on \( U \). So \( \dim(\mathcal{S}(\phi)) = 1 \).

Now that we have a holomorphic map \( \phi \) from \( X \) to a curve \( C \), take its Stein factorization

\[
X \xrightarrow{\bar{\psi}} D \xrightarrow{\varphi} C.
\]

Since \( g \) is finite, \( D \) is a (compact complex) curve. Moreover, since \( X \) is a manifold, \( D \) is also normal, hence smooth. By definition of the Stein factorization, we thus get a fibration of \( X \) over \( D \). We denote by \( S \) its set of critical values.

It only remains to show that the forms \( \omega_1, \ldots, \omega_g \) come from \( D \). Let \( j \) be an integer in \( \{1, \ldots, g\} \) and consider the open set \( V_j \subset D \) defined by \( V_j = \psi(U_j) - S \). By the proper mapping theorem \( D - \psi(U_j) \) is a nowhere dense analytic set in \( D \), hence a finite number of points. If \( y \) is a point in \( V_j \), we consider the connected fiber \( F_y = \psi^{-1}(y) \). We want to define \( \omega_j(y)(Y) \) by \( \omega_j(x)(Z) \), where \( x \) is in \( F_y \cap U_j \) and \( Z \) is in the tangent space of \( X \) at \( x \) and is sent to \( Y \) by \( d\psi_x \).

Since \( y \) is not a critical value, \( \psi \) is locally a trivial fibration and we can assume

- \( U_i \) is a domain in \( \mathbb{C}^n \) with coordinates \( (z_1, \ldots, z_n) \);
- \( D \) is a domain in \( \mathbb{C} \) with coordinate \( z_1 \);
- \( \psi \) is the projection on the first factor.

In these coordinates, \( \omega_i \) can be written \( \sum_k f_k dz_k \). Since the foliation \( F \) is defined on \( U_j \) by \( \omega_j = 0 \) and corresponds locally to the fiber of \( \psi \), we have that \( f_k = 0 \) for \( k \geq 2 \). Moreover, since \( \omega_j \) is closed, \( f_1 \) can be seen as a function of \( z_1 : \omega_j = f_1(z_1) dz_1 \). Take \( y \) in \( V_j \) and \( Y \) in \( T_y D \). A point \( x \) in \( F_y \cap U_j \) can be written \( x = (y, z_2, \ldots, z_n) \) and a tangent vector \( Z \) in \( T_x \) is sent to \( X \) by \( d\psi_x \) if and only if it is written \( X = (Y, Z_2, \ldots, Z_n) \) where \( Z_k \) is arbitrary for \( k \geq 2 \). Hence

\[
\omega_j(x)(X) = f_1(y)(Y),
\]

i.e. \( \omega_j(x)(X) \) is independent on the choices made when \( x \) stays in a fixed connected component of \( F_y \cap U_j \). But since \( F_y \) is connected and \( F_y \cap U_j \) is dense in \( F_y \), the values in different connected components coincide and we can define \( \omega_j(y)(Y) \).

From the equation (7.1), it is clear that \( \omega_j \) is a holomorphic 1-form on \( V_j \). We can extend \( \omega_j \) to a holomorphic 1-form on all \( D = V_j \cup \{ \text{discrete set of points} \} \). Indeed, by the Riemann extension theorem, it is sufficient to prove that \( \omega_j \) is locally bounded. Since, \( \omega_j \) is defined by use of \( \omega_j \), which is a holomorphic 1-form on the compact manifold \( X \), this concludes the proof of the theorem.
An analogous statement for the complex cohomology was first proven in [Cat91].

**Theorem 7.5** (Catanese). Let \( X \) be a compact Kähler manifold and \( V \subset H^1(X, \mathbb{C}) \) a \( g \)-dimensional isotropic subspace with \( g \geq 2 \). Then there exists a fibration \( f \) to a curve \( C \) such that \( V \subset f^*H^1(C, \mathbb{C}) \). Necessarily the genus of the curve is at least \( g \).

**Proof.** Let \( \phi_1, \ldots, \phi_g \) be a basis of \( V \) and write \( \phi_i = \omega_i + \overline{\eta}_i \) with \( \omega_i, \eta_j \in H^0(X, \Omega_X) \). Denote by \( U \) and \( W \) the span of the \( \omega_i \) and of the \( \eta_j \). The assumption \( \phi_i \wedge \phi_j = 0 \) and the bidegree decomposition leads to the following equations:

\[
\begin{align*}
\omega_i \wedge \omega_j &= 0; \\
\eta_i \wedge \eta_j &= 0; \\
(\omega_i \wedge \eta_j) + (\eta_i \wedge \omega_j) &= 0.
\end{align*}
\]

(7.2) \hspace{1cm} (7.3) \hspace{1cm} (7.4)

In particular \( U \) and \( W \) are isotropic subspaces in \( H^0(X, \Omega_X) \).

- If their dimensions are both at least 2, then we can apply Castelnuovo-de Franchis theorem twice and obtain two fibrations from \( X \) to curves of genuses at least 2. By construction \( U \) and \( W \) are in the pull-back of the product map \( \phi : X \to C \times C' \), hence \( V \) is also in the pull-back. By the Künneth formula, \( H^*(C \times C', \mathbb{C}) = H^*(C, \mathbb{C}) \otimes H^*(C', \mathbb{C}) \). So, if \( \alpha_i \) (resp. \( \beta_j \)) are holomorphic 1-forms on \( C \) (resp. \( C' \)) that pull-back to \( \omega_i \) (resp. \( \eta_j \)) then \( (\alpha_i \wedge \beta_j) + (\beta_i \wedge \alpha_j) \) has to be non-zero. Since \( \phi \) is holomorphic between Kähler manifolds, it is injective in cohomology; so we would obtain a contradiction with the third equation in (7.2) if \( \phi \) were surjective.

By the proper mapping theorem, the image of \( \phi \) is an analytic set in \( C \times C' \); by what precedes it is a (possibly singular) complex curve \( C'' \). The intermediate space \( D \) in a Stein factorization of \( \phi : X \to C'' \) is a smooth curve since \( X \) is a complex manifold. Since the cohomology of \( C'' \) goes through the cohomology of \( D \), this concludes this case.

- If the dimension of \( U \) (by symmetry) is 1, note that the dimension of \( W \) is at least 2 since \( V \) is not one-dimensional. By a change of basis of \( V \), we can assume that \( \omega_1 = 0 \) for \( i \) from 2 to \( g \). For \( j \) from 2 to \( g \), the third equation in (7.2) implies \( \omega_1 \wedge \overline{\eta}_j = 0 \). If \( \omega_1 \wedge \eta_j \) were not 0, then \( \omega_1 \wedge \eta_j \wedge \omega_1 \wedge \overline{\eta}_j \) neither by (3.3), which is a contradiction. Hence \( \omega_1 \wedge \eta_j = 0 \). We also have \( \omega_1 \wedge \eta_1 = 0 \). Indeed, choose a \( j \geq 2 \) such that \( \eta_j \) is not zero. Then \( \eta_1 \wedge \eta_j = \omega_1 \wedge \eta_j = 0 \) implies that \( \omega_1 \wedge \eta_1 = 0 \) wherever \( \eta_j \) does not cancel. Since the set where \( \eta_j \) cancels has no interior, we have in fact the equality everywhere.

This proves that the space \( U + W \) is isotropic. Since it is of dimension at least 2, one can apply Castelnuovo-de Franchis theorem and obtain a holomorphic map from \( X \) to a curve, whose pull-back contains \( V \).

\[\Box\]

**Definition 7.3** (Notation). We denote by \( g_\mathbb{R}(X) \) (resp. \( g_\mathbb{C}(X) \)) the maximal real (resp. complex) dimension of an isotropic subspace for real (resp. complex) cohomology and by \( g_h(X) \) the maximal complex dimension of an isotropic subspace for holomorphic 1-forms.

**Proposition 7.6.** We have the equalities \( g_\mathbb{R}(X) = g_\mathbb{C}(X) = g_h(X) \). This common quantity is called the genus of \( X \) and is denoted by \( g(X) \).
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Proof. First note that, by tensoring with \( \mathbb{C} \) an isotropic subspace in real cohomology, we get an isotropic subspace of the same dimension in complex cohomology. Moreover, an isotropic subspace in \( H^0(X, \Omega_X) \) is also an isotropic subspace in \( H^1(X, \mathbb{C}) \). So we have the two inequalities:

\[
g_\mathbb{C}(X) \leq g(X) \quad \text{and} \quad g_\mathbb{R}(X) \leq g_\mathbb{C}(X).
\]

By Castelnuovo theorem, if \( g_\mathbb{C}(X) \geq 2 \), there exists a fibration \( f \) of \( X \) over a curve \( C \) of genus \( \geq g_\mathbb{C}(X) \). Consider a basis \((\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)\) for the group of singular homology \( H_1(C, \mathbb{Z}) \) such that the only curves intersecting are \( \alpha_i \) and \( \beta_i \). Denote by \( \lambda_1, \ldots, \lambda_g \) the Poincaré duals of \( \alpha_1, \ldots, \alpha_g \). Then, for every \( i \neq j \), the cup-product between \( \lambda_i \) and \( \lambda_j \) is the Poincaré dual of the homology class of the intersection of \( \alpha_i \) and \( \alpha_j \). Since they don’t intersect, \( \lambda_i \cup \lambda_j = 0 \). Hence \( g_\mathbb{R}(C) \geq g \). Moreover the space of holomorphic 1-forms in \( C \) is of dimension \( g \) and is isotropic since \( H^2(C, \mathbb{C}) = H^{1,1}(C) \). This gives the inequality \( g_\mathbb{R}(C) \geq g \).

The pullback by \( f^* \) of these isotropic subspaces gives isotropic subspaces in \( H^1(X, \mathbb{R}) \) and \( H^0(X, \Omega_X) \) of dimension \( g \) since \( f^* \) is injective on \( H^1 \). So we get the other inequalities \( g_\mathbb{R}(X) \geq g_\mathbb{C}(X) \), \( g_\mathbb{R}(X) \geq g_\mathbb{C}(X) \) if \( g_\mathbb{C}(X) \geq 2 \).

The case \( g_\mathbb{C}(X) = 0 \) means that the first Betti number is 0 and so the equalities are obvious. Finally, \( g_\mathbb{R}(X) = 1 \), the first Betti number of \( X \) is not zero and any line in real cohomology or in holomorphic 1-forms is an isotropic subspace of dimension 1. This concludes the proof.

We have discussed two topological invariants, constructed from the cohomology of \( X \): \( a(X) \) which corresponds geometrically to the complex dimension of the Albanese image of \( X \) and \( g(X) \). Looking at the definitions in terms of the cohomology of \( X \), it is clear that \( a(X) \) does not depend on the ground field of characteristic 0 since the canonical map \( \Lambda^1 H^1(X, \mathbb{F}) \rightarrow H^1(X, \mathbb{F}) \) is zero if and only if it is zero for \( \mathbb{F} = \mathbb{Q} \).

But it is false that \( g(X) \) is independent on the ground field for linear algebra reasons, contrary to what is said implicitly after the definition of the genus, on page 24 in [ABC+96].

One can make this precise by the following:

Proposition 7.7. There exists \( n, m \in \mathbb{N} \) and a skew-symmetric \( \mathbb{Z} \)-bilinear map \( \alpha \) from \( \mathbb{Z}^n \times \mathbb{Z}^n \) to \( \mathbb{Z}^m \) such that the maximal complex dimension of an isotropic subspace in \( \mathbb{C}^n \) (for \( \alpha_{\mathbb{C}} \)) is strictly larger than the maximal real dimension of an isotropic subspace in \( \mathbb{R}^n \) (for \( \alpha_{\mathbb{R}} \)).

Proof. This construction was given to me by David Speyer on the website MathOverflow.

The idea is the following: consider four \( 2 \)-planes \( V_i \) in \( \mathbb{R}^4 \) in general position. There exists two complex-conjugate \( 2 \)-planes \( W \) and \( W' \) in \( \mathbb{C}^4 \) that intersect the four planes \( (V_i)_{\mathbb{C}} \) non-trivially. Indeed, it is equivalent to count the number of lines intersecting four given lines in general position in \( \mathbb{C}P^3 \) and the result is known by Schubert calculus, see [KLT2].

Denote by \( \omega_i \) four skew-symmetric forms whose kernels are exactly the \( V_i \). We claim that a \( 2 \)-plane \( Z \) is isotropic for the skew-symmetric bilinear map \( \alpha = (\omega_1, \omega_2, \omega_3, \omega_4) \) if and only if it intersects the \( V_i \) non-trivially.

Proof of the claim. If \( Z \) is isotropic for \( \alpha \), \( Z \) is isotropic for all \( \omega_i \). If \( Z \) intersected a \( V_i \) trivially, \( \omega_i \) would be zero since it is zero on \( Z \times Z \), on \( V_i \times Z \) an on \( Z \times V_i \), a contradiction.

Conversely, if \( Z \) is the span of \( x_i \) and \( y_i \) with \( x_i \in V_i \) then \( \omega_i(x_i, y_i) = 0 \) because \( x_i \in \ker(\omega_i) \) and \( \omega_i(x_i, x_i) = \omega_i(y_i, y_i) \) by skew-symmetry. So \( Z \) is \( \omega_i \)-isotropic for all \( i \), hence \( \alpha \)-isotropic.

Hence \( W \) and \( W' \) are the only maximal isotropic subspaces for \( \alpha \). If they are not defined over the reals, then we have that \( g_\mathbb{C}_i(X) = 2 \) but \( g_\mathbb{R}(X) = 1 \), with the above notations.

An explicit example is \( V_1 = \text{Span}_{\mathbb{R}}(e_1, e_2) \), \( V_2 = \text{Span}_{\mathbb{R}}(e_3, e_4) \), \( V_3 = \text{Span}_{\mathbb{R}}(e_1 + e_3, e_2 + e_4) \) and \( V_4 = \text{Span}_{\mathbb{R}}(e_1 + e_4, e_2 - e_3) \); then \( W \) and \( W' \) are only defined over \( \mathbb{C} \): \( W = \text{Span}_{\mathbb{C}}(e_1 + ie_2, e_3 + ie_4) \) and \( W' = \text{Span}_{\mathbb{C}}(e_1 - ie_2, e_3 - ie_4) \).

Since in this example, the \( \omega_i \) can be defined over the integers, this proves the proposition.
Remark 7.4.

- A bilinear map like in the above proposition can always be seen as the cup-product in \( H^1(X, \mathbb{Z}) \) of some CW-complex \( X \); so this is not only a linear algebra construction.
- Since we have equality of the real and complex genuses in the case of Kähler manifolds, this gives a restriction on the cohomology of a finitely presentable group for being Kähler.
- The proof that we give of the equality of the genera uses fibration over complex curves, hence is quite indirect. Maybe one could find a linear algebra proof, using the bidegree decomposition and maybe the Hard Lefschetz theorem.

7.3 Siu-Beauville theorem

The above discussion shows a deep interaction between the geometry of a Kähler manifold (whether it fibers or not above a complex curve) and the forms on \( X \) (complex cohomology or holomorphic 1-forms). In fact, this question of fibration over a curve can already be answered at the level of the fundamental group of \( X \).

Theorem 7.8 (Siu-Beauville). Let \( X \) be a compact Kähler manifold. The following statements are equivalent:

1. \( \pi_1(X) \) surjects onto \( \pi_1(C_g) \) where \( C_g \) is a curve of genus \( g \), for some \( g \geq 2 \).
2. \( \pi_1(X) \) surjects onto \( F_g \) where \( F_g \) is the free group on \( g \) generators, for some \( g \geq 2 \).
3. The genus of \( X \) is at least 2.
4. \( X \) fibers over a curve of genus at least 2.

Proof.

- 1 \( \implies \) 2: The standard presentation of \( G := \pi_1(C_g) \) is

\[
G = \langle a_1, \ldots, a_g, b_1, \ldots, b_g | [a_1, b_1] \ldots [a_g, b_g] \rangle.
\]

Adding new relations \( b_1 = \ldots = b_g = 0 \), one clearly obtains the free group on \( g \) generators.

- 2 \( \implies \) 3: Let \( \phi \) be a surjection, as in (2). Then there is a map \( B\phi \) from \( B\pi_1(X) \) to \( BF_g \) which is injective on \( H^1 \). Since the canonical map \( c_X : X \to B\pi_1(X) \) is an isomorphism on \( H^1 \), this gives an injection from \( H^1(BF_g) \) to \( H^1(X) \). But \( BF_g \) is a wedge of \( g \) circles, hence has no \( H^2 \). So \( H^1(BF_g) \) is isotropic of dimension \( g \), hence also its image in \( H^1(X) \).

- 3 \( \implies \) 4: This is the Castelnuovo-de Franchis theorem.

- 4 \( \implies \) 1: This is lemma 7.2.

We sum up the different results of the last two sections as a corollary, where the assertions of maximality come from the proofs of the Catanese and Siu-Beauville theorems.

Corollary 7.9. The genus \( g(X) \) of a Kähler manifold \( X \) is an invariant of the fundamental group of \( X \). Assume that it is at least 2. Then it can be seen as:
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- The maximal dimension of an isotropic subspace in real or complex cohomology or in holomorphic 1-forms.
- The maximal $g$ such that $\pi_1(X)$ surjects onto the fundamental group of a curve of genus $g$.
- The maximal genus of a curve on which $X$ fibers.
Chapter 8

One-dimensional Albanese image

In this chapter, we concentrate on Kähler manifolds with 1-dimensional Albanese image. We show that in this case the image is smooth and that the Albanese map is a fibration. Moreover, we prove that having a 1-dimensional Albanese image is a property of the fundamental group. In the last section, we apply these techniques to determine which Kähler groups are also 3-manifolds groups.

8.1 Smooth image and Albanese map with connected fibers

Theorem 8.1. For a smooth curve $C$ of genus $g > 0$, $\text{alb}_C$ is an embedding.

Proof.

- We first show that $\text{alb}_C$ is an immersion. Assume by contradiction that $\text{alb}_C$ is not an immersion at $p \in C$. Then since a global holomorphic section $\omega \in H^0(C, K)$ is of the form $\text{alb}^*(\alpha)$ with $\alpha \in H^0(\text{Alb}(C), \Omega_{\text{Alb}(C)})$ one has $\omega(p) = 0$ for every such $\omega$. So every holomorphic section of the canonical bundle $K$ vanishes at $p$, which implies that $\dim H^0(K \otimes \mathcal{O}(-p)) = \dim H^0(K) = g$.

By Serre duality and the Riemann-Roch formula (see [Huy04]) for line bundles over curves, one has

$$\dim H^0(\mathcal{O}(p)) = \chi(C, \mathcal{O}(p)) + \dim H^1(\mathcal{O}(p)) = 1 + (1 - g) + g = 2.$$  

Denote by $\sigma$ and $\tau$ two non-collinear sections in $H^0(\mathcal{O}(p))$. Then one can consider the ratio $f = \frac{\sigma}{\tau}$ as a holomorphic map from $C$ to $\mathbb{C}P^1$.

- If both sections vanish at the same point $q$ then $f$ does not take the value 0 so is constant. This contradicts the non-collinearity of $\sigma$ and $\tau$.

- If $\sigma$ but not $\tau$ vanishes at $q$ then looking at the fiber of $0 \in \mathbb{C}P^1$ one obtains that the degree of the map $f$ is 1. This contradicts $g(C) > 0$.

This proves that $\text{alb}_C$ is an immersion.

- To show the injectivity, it is useful to recall the definition of the Abel-Jacobi map for a curve. The exponential sequence (see [Huy04])

$$0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$$

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induces a long exact sequence in cohomology
\[ \cdots \rightarrow H^1(C, \mathbb{Z}) \rightarrow H^1(C, \mathcal{O}_C) \rightarrow H^1(C, \mathcal{O}_C^*) \rightarrow H^2(C, \mathbb{Z}) \rightarrow \cdots. \]

The quotient \( \frac{H^1(C, \mathcal{O}_C)}{H^1(C, \mathbb{Z})} \) is the Albanese torus of \( C \) (see the definition 3.3 with \( n = 1 \)) and can be identified with the kernel of the map \( H^1(C, \mathcal{O}_C^*) \rightarrow H^2(C, \mathbb{Z}) \). For a curve \( C \), \( H^2(C, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \) and the previous map gives the degree in \( \mathbb{Z} \) of the (isomorphism classes of) line bundles in \( H^1(C, \mathcal{O}_C^*) \). In particular, one can define a map
\[ \psi : C \rightarrow \text{Alb}(X) \]
\[ x \mapsto \mathcal{O}(x - x_0), \]
where \( x_0 \) is a fixed point in \( C \) and \( \mathcal{O}(x - x_0) \) is the line bundle associated to the divisor \( x - x_0 \). This is the Abel-Jacobi map.

If \( \text{alb}_C \) were not injective, then since \( \psi \) is a map from \( C \) to a torus, it factorizes through \( \text{alb}_C \), hence \( \psi \) would not be injective. If \( \psi(p) = \psi(q) \), then \( \mathcal{O}(p - x_0) \) and \( \mathcal{O}(q - x_0) \) are isomorphic, that is \( \mathcal{O}(p) \) and \( \mathcal{O}(q) \) are isomorphic. Hence there exists a line bundle \( L \) and \( s_1, s_2 \) two sections of \( L \) such that \( s_1 \) vanishes only at \( p \) and \( s_2 \) vanishes only ay \( q \) (and with multiplicity one). We define a surjective holomorphic map
\[ \theta : C \rightarrow \mathbb{C}P^1 \]
\[ x \mapsto [s_1(x) : s_2(x)]. \]

The fiber over \([0 : 1]\) contains only \( p \) and \([0 : 1]\) is not a critical value since \( s_1 \) vanishes at \( y \) with multiplicity one. Hence \( \theta \) is a degree one-map and the fiber on any non-critical value contains only one point. Since \( \theta \) would be non-injective in a neighbourhood of any critical point, \( \theta \) is in fact an immersion. Hence \( \theta \) is a biholomorphism between \( C \) and \( \mathbb{C}P^1 \), in contradiction to \( g(C) \geq 1 \).

Since \( \text{alb}_C \) is a proper injective immersion, it is an embedding. \( \square \)

**Remark 8.1.** The Albanese map and the Abel-Jacobi map are in fact the same when one identifies the two models of the Albanese torus and chooses the same point \( x_0 \) in both definitions.

**Theorem 8.2.** Let \( X \) be a compact Kähler manifold. If \( a(X) = 1 \) then \( \text{alb}_X(X) = C \) is a smooth curve embedded in \( \text{Alb}(X) \). Moreover \( \text{alb}_X \) has connected fibers.

**Proof.** Consider the following commutative diagram:
\[ \begin{array}{ccc}
X & \xrightarrow{\text{alb}_X} & C \\
\downarrow{\text{alb}_X} & & \downarrow{\text{alb}(\bar{C})} \\
\bar{C} & \xrightarrow{\bar{a}} & \text{Alb}(\bar{C}) \\
\end{array} \]

In this diagram,
- \( \text{alb}_X \) is the corestriction of the Albanese map of \( X \) to its image and \( \iota \) is the inclusion of \( C \) in \( \text{Alb}(X) \);
- \( (\bar{C}, j) \) is the normalization of the curve \( C \);
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- $a$ is obtained by the universal property of $\text{Alb}(\overline{C})$ applied to $\iota \circ j$;
- $b$ is obtained by the universal property of $\text{Alb}(X)$ applied to $\text{alb}_C \circ \text{alb}_X$ (see 1.11 for the existence of $\text{alb}_X$).

The vertical maps on the right are inverse isomorphisms. Indeed since the diagram is commutative one has

$$a \circ b \circ \iota \circ \text{alb}_X = a \circ \text{alb}_C \circ \overline{\text{alb}}_X = \iota \circ j \circ \overline{\text{alb}}_X = \iota \circ \text{alb}_X.$$  

So by uniqueness, $a \circ b = \text{id}_{\text{Alb}(X)}$.

Similarly

$$b \circ a \circ \text{alb}_C \circ \overline{\text{alb}}_X = b \circ \iota \circ j \circ \overline{\text{alb}}_X = b \circ \iota \circ \text{alb}_X = \text{alb}_C \circ \overline{\text{alb}}_X.$$  

Since $\overline{\text{alb}}_X$ is surjective this implies $b \circ a \circ \text{alb}_C = \text{alb}_C$ and by uniqueness $b \circ a = \text{id}_{\text{Alb}(\overline{C})}$. With the identification $\text{Alb}(X) \approx \text{Alb}(\overline{C})$, the images of the embeddings $\text{alb}_C$ and $\iota$ are the same, which by the previous lemma means that $C = \overline{C}$.

We claim that $\text{alb}_X$ has connected fibers. For this consider the following commutative diagram with the Stein factorisation of $\text{alb}_X$.

Note that since $X$ is smooth, $D$ is smooth too and we can speak of its Albanese torus. The map $a$ is given by considering $\text{alb}_C \circ h$ and the map $b$ by considering $\text{alb}_D \circ g$. As before, using that $g$ is surjective, one can see that $a$ and $b$ are inverse isomorphisms. This implies that $\text{Alb}(C) = \text{Alb}(D)$ and that $h$ is the identity of $C$ since both $\text{alb}_C$ and $\text{alb}_D$ are embeddings. Hence we get $C = D$ and $\text{alb}_X$ has connected fibers. \hfill $\square$

8.2 A characterization

The Albanese dimension of $X$ is a topological invariant of $X$, but not an invariant of the fundamental group alone. However, the property of having Albanese dimension one is a property of the fundamental group.

We want to give several different characterizations of Kähler manifolds with one-dimensional Albanese image. One has to distinguish the two cases where the image is of genus one, or is of genus at least two. We first give a precise definition of the cup-product for a topological space $X$ or for a group $G$.

**Definition 8.1.** The cup-product for a topological space $X$ (resp. a group $G$) is the natural application $\Lambda^2 H^1(X, \mathbb{R}) \to H^2(X, \mathbb{R})$ (resp. $\Lambda^2 H^1(G, \mathbb{R}) \to H^2(G, \mathbb{R})$) corestricted to its image. It is only determined by the dimension of the $H^1$ and the kernel of the cup-product.
Remark 8.2. There is naturally a notion of isomorphic cup-products. By the following lemma, the theorem (2.8) exactly says that a CW-complex $X$ and its fundamental group $\pi_1(X)$ have the same cup-product, although $H^2(G, \mathbb{R})$ and $H^2(X, \mathbb{R})$ are generally not isomorphic.

Lemma 8.3. If $f$ is a continuous map between topological spaces $X$ and $Y$ and induces an isomorphism on $H^1$, and a monomorphism on $H^2$, then $X$ and $Y$ have the same cup-product over $\mathbb{R}$.

Proof. This is obvious since $H^1(Y, \mathbb{R})$ is isomorphic to $H^1(X, \mathbb{R})$ by $f^*$ and a cup-product $\alpha \cup \beta$ is 0 in $H^2(Y, \mathbb{R})$ if and only if $f^*(\alpha) \cup_X f^*(\beta)$ is 0 in $H^2(X, \mathbb{R})$.

The following theorem deals with the case of 1-dimensional Albanese image of genus 1.

Theorem 8.4. For a compact Kähler manifold $X$ the following conditions are equivalent:

1. $b_1(X) = 2$;
2. The Albanese image of $X$ is an elliptic curve;
3. $X$ admits a fibration to an elliptic curve which induces an isomorphism on $H^1$;
4. The cup-products for $\pi_1(X)$ and $\mathbb{Z}^2$ are isomorphic.

Proof. (1) $\Rightarrow$ (2): If the first Betti number of $X$ is 2, $\text{Alb}(X)$ is an elliptic curve. Since $\text{alb}_X$ is non-constant, it is surjective.

(2) $\Rightarrow$ (3): If (2) holds, then $\text{alb}_X$ has all the properties required for $f$, by the preceding section.

(3) $\Rightarrow$ (1): The first Betti number of an elliptic curve is 2 and $f$ induces an isomorphism on $H^1$;

(2) $\Rightarrow$ (4): Denote by $T$ the Albanese image of $X$. The Albanese map of $X$ is surjective, hence a monomorphism on cohomology, in particular on $H^2$. Because it is also an isomorphism on $H^1$, $\pi_1(X)$ and $T$ have the same cup-product, which is also the cup-product for $\pi(T) = \mathbb{Z}^2$.

(4) $\Rightarrow$ (1): The first Betti number of $\mathbb{Z}^2$ is 2.

Now we treat the case of one-dimensional Albanese images of higher genus.

Theorem 8.5. For a compact Kähler manifold $X$ the following conditions are equivalent:

1. The Albanese image of $X$ is a curve $C$ with $g(C) \geq 2$;
2. $X$ admits a holomorphic map to a curve of genus $\geq 2$ inducing an isomorphism on $H^1$;
3. $X$ admits a fibration to a curve of genus $\geq 2$ inducing an isomorphism on $H^1$;
4. $\pi_1(X)$ admits a surjective homomorphism $\varphi$ onto some $\pi_1(C_g)$ with $g \geq 2$, such that $\varphi^*$ is an isomorphism on $H^1$;
5. Every element of $H^1(X)$ is contained in a two-dimensional isotropic subspace for the cup product $H^1 \times H^1 \rightarrow H^2$;
6. $b_1(X) > 2$ and the cup product $H^1 \times H^1 \rightarrow H^2$ has one-dimensional image;
7. The cup-product for $\pi_1(X)$ is isomorphic to that of some $\pi_1(C_g)$ with $g \geq 2$;
8. $b_1(X) > 2$ and the cup-length of holomorphic 1-forms is 1, i.e. every wedge product of holomorphic 1-forms is trivial.
Proof. We prove the following implications:

\[ (1) \Rightarrow (3): \] The Albanese map has these properties.

\[ (3) \Rightarrow (4): \] If \( f \) is a map as in (3), it induces \( \phi = f_* \) from \( \pi_1(X) \) to \( \pi_1(C) \) where \( C \) is a curve of genus \( \geq 2 \). Because \( f \) is surjective with connected fibers, \( \phi \) is surjective. Denote by \( K \) the Eilenberg-Mac Lane space \( K(\pi_1(X), 1) \) and by \( c_X \) the tautological map from \( X \) to \( K \). Then \( \phi \) induces a continuous map \( B\phi \) from \( K \) to \( C \) (because \( C \) is aspherical, as a quotient of the hyperbolic plane) with \( (B\phi)_* = \phi \). As \( C \) is aspherical \( f \) factors to \( \tilde{f} \) from \( K \) to \( C \) such that \( \tilde{f} \circ c_X = f \). Now \( \tilde{f} \) and \( B\phi \) induce the same map \( \varphi \) on fundamental groups, so they are homotopic and \( f^* = c_X^* \circ B\phi^* \) on \( H^1 \). As \( f^* \) and \( c_X^* \) are both isomorphisms on \( H^1 \), \( B\phi^* \) too. But this is just \( \phi^* \), by definition.

\[ (4) \Rightarrow (7): \] Denote by \( B\phi \) the continuous map from \( K = K(\pi_1(X), 1) \) to \( C \) induced by \( \phi \). By hypothesis \( f = B\phi \circ c_X \) induces an isomorphism on \( H^1 \). It also induces a monomorphism on \( H^2 \). Indeed, if \( f^* \) is not a monomorphism on \( H^2 \), it is the zero map since \( H^2(C) \) is one-dimensional. Take a holomorphic form \( \alpha \) on \( X \); it can be seen as an element in \( H^1(X, \mathbb{C}) \). By (3.3), \( \alpha \wedge \bar{\alpha} \) is a non-zero class in \( H^2(X, \mathbb{R}) \). Since \( f^* \) is an isomorphism on \( H^1 \), this class comes from \( H^2(C) \) and \( f^* \) is a monomorphism on \( H^2 \). This proves that \( X \) and \( C \) as the same cup-product.

\[ (3) \Rightarrow (2): \] This is obvious.

\[ (2) \Rightarrow (7): \] By hypothesis, \( f^* \) is an isomorphism on \( H^1 \). It is also a monomorphism on \( H^2 \) since \( f \) is surjective.

\[ (7) \Rightarrow (6): \] The cup-products of \( X \) and \( C \) are the same and the assertions in (6) are true when \( X = C \).

\[ (6) \Rightarrow (5): \] Take \( \alpha \) in \( H^1(X) \) Then the cup product with \( \alpha \) from \( H^1(X) \) to \( H^2(X) \) has at most one-dimensional image. As \( b_1(X) > 2 \) its kernel contains a \( \beta \) non-collinear with \( \alpha \). The subspace generated by \( \alpha \) and \( \beta \) is isotropic for the cup product.

\[ (5) \Rightarrow (7): \] This is the most delicate part of the proof. Let \( U \) be an isotropic subspace of \( H^1(X, \mathbb{C}) \) such that \( \dim U \geq 2 \). By theorem (7.5), there exists a fibration \( f \) from \( X \) to a curve \( C \) such that \( U \) is contained in \( f^*(H^1(X; \mathbb{C})) \). Condition (5) implies that every element of \( H^1(X, \mathbb{C}) \) lies in such an isotropic subspace \( U \). Hence we get

\[ H^1(X, \mathbb{C}) = \bigcup_f f^*(H^1(C_f; \mathbb{C})) , \]

where \( f \) evolves in all fibrations from \( X \) to a curve \( C_f \) of genus \( \geq 2 \). Note that each \( f^* \) is induced by a map from \( H^1(C, \mathbb{Q}) \) to \( H^1(X, \mathbb{Q}) \) so there are only countably many such maps. Hence the equality above implies that at least one of the \( f^*(H^1(C_f; \mathbb{C})) \) equals \( H^1(X, \mathbb{C}) \) because a \( \mathbb{C} \)-vector space is not the union of countably many proper subspaces by Baire’s theory. Such an \( f \) is an isomorphism on \( H^1 \) and a monomorphism on \( H^2 \).

\[ (6) \Rightarrow (8): \] The decomposition \( H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus H^0(X, \Omega_X) \) induces a decomposition \( H^1(X, \mathbb{C}) \cup H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \cup H^0(X, \Omega_X) \oplus H^0(X, \Omega_X) \cup H^0(X, \Omega_X) \cup H^0(X, \Omega_X) \cup H^0(X, \Omega_X) \) for the image of the cup-product. The second term is not zero since it contains the non-zero class \( \alpha \wedge \bar{\alpha} \) for any non-zero holomorphic form \( \alpha \). Hence the first term is zero.

\[ (8) \Rightarrow (1): \] The cup-length on holomorphic 1-forms is the dimension of the Albanese image of \( X \). Hence the Albanese image of \( X \) is a (smooth) curve, necessarily of genus \( \geq 2 \) because the Albanese map is an isomorphism on \( H^1 \) and \( b_1(X) > 2 \).

\[ \square \]

Remark 8.3. For a Kähler group \( G \) and an integer \( g \geq 1 \), we will say that \( G \) has Albanese image \( C_g \) if \( G \) satisfies condition 4 of the theorem.
8.3 Kähler groups and 3-manifolds groups

An application of this study of the Albanese map and of the fibrations over complex curves is the proof of a theorem already discussed in (6.4).

Theorem 8.6. Let $G$ be a group which is both the fundamental group of a closed 3-manifold and a Kähler group. Then $G$ is finite.

This theorem was first proved by Dimca and Suciu in [DS09]. In [Kot11], Kotschick gives another proof of it, further using what is known about 3-manifolds. We recall some of these facts.

Definition 8.2.

- A connected 3-manifold $M$ is prime if $M$ is not homeomorphic to $S^3$ and if a connected sum decomposition $M = P\#Q$ implies that $P$ or $Q$ is homeomorphic to $S^3$.

- A connected 3-manifold $M$ is irreducible if every embedded 2-sphere $S^2$ in $M$ bounds an embedded ball $B^3$.

Fact 8.7 ([Hat07], p.6). Let $M$ be a compact, connected and oriented 3-manifold. Then there is a decomposition $M = P_1\#\ldots\#P_n$, where each $P_i$ is a prime manifold. Moreover the decomposition is unique up to order.

Fact 8.8 ([Hat07], p.6). The only orientable prime manifold which is not irreducible is $S^1 \times S^2$.

Fact 8.9. If $M$ is irreducible then $\pi_2(M) = 0$.

Sketch of the proof. If $\pi_2(M)$ is not trivial then the sphere theorem (see [Pap57]) says that there exists an embedded 2-sphere in $M$ representing a non-trivial element of $\pi_2(M)$. In particular this sphere does not bound an embedded ball, hence $M$ is not irreducible.

Lemma 8.10. If $M$ is an irreducible manifold with $\pi_1(M)$ infinite, then $M$ is aspherical.

Proof. Consider $\tilde{M}$ the universal covering space of $M$. By 8.9, $\pi_1(\tilde{M}) = \pi_2(\tilde{M}) = 0$. Moreover since $\pi_1(M)$ is infinite, $\tilde{M}$ is a non-compact 3-manifold and $H_i(\tilde{M}) = 0$ for $i \geq 3$.

Since $\tilde{M}$ is 2-connected, the Hurewicz theorem states that there is an epimorphism $H_3(\tilde{M}) \to \pi_3(\tilde{M})$. So $\pi_3(\tilde{M})$ is trivial and, by iterating, all the higher homotopy groups of $\tilde{M}$ are trivial. So $M$ is aspherical.

Proposition 8.11. Let $M$ be a compact oriented aspherical 3-manifold with infinite fundamental group $G$ and vanishing first Betti number. Then $M$ has a finite covering with positive first Betti number or $M$ is hyperbolic.

Remark 8.4. We will not prove this difficult proposition and refer to [Kot11]. Nevertheless, we can notice that the work of Perelman is used in the second part of the alternative.

With all these facts about the world of 3-manifolds, we can prove theorem 8.6. We begin with a lemma which is interesting for itself.

Lemma 8.12. If $G$ is a group with positive first Betti number and whose real cohomology algebra satisfies 3-dimensional oriented Poincaré duality, then $G$ is not Kähler.
8.3. KÄHLER GROUPS AND 3-MANIFOLDS GROUPS

Proof. Suppose the contrary and denote by $X$ a compact Kähler manifold with fundamental group $G$. Consider the Albanese map $\text{alb}_X : X \to \text{Alb}(X) = T^{b_1}(X)$. Since $T^{b_1}(X)$ is aspherical, this Albanese map factorizes as:

$$X \xrightarrow{c_X} BG \xrightarrow{\alpha} T^{b_1}(X).$$

Hence $\text{alb}_X^* = c_X^* \circ \alpha^*$. Since $H^i(BG; \mathbb{R}) = 0$ for $i > 3$, $\text{alb}_X^*$ is zero in degrees $> 3$. If the Albanese dimension of $X$ were $\geq 2$, then the cup-length of $X$ in real cohomology would be at least 4. But since $\text{alb}_X^*$ is an isomorphism on $H^1$, a non-zero class in the image of the natural map $\Lambda^4 H^1(X; \mathbb{R}) \to H^4(X; \mathbb{R})$ would come from $T^{b_1}(X)$ by $\text{alb}_X^*$, which is a contradiction. Hence the Albanese image of $X$ is a complex curve $C$.

The above factorization corestricts to $C = \text{alb}_X(X)$. Since $\text{alb}_X^*$ is non trivial in degree 2, $\alpha^* : H^2(C, \mathbb{R}) \to H^2(BG, \mathbb{R})$ is also non trivial. But $\alpha$ is an isomorphism on $H^1$ since it is true for $\text{alb}_X$ and $c_X$. Given a non trivial class $\alpha \in \alpha^*(H^2(C, \mathbb{R})) \subset H^2(BG, \mathbb{R})$, 3-dimensional oriented Poincaré duality implies that there exists $\beta \in H^1(BG, \mathbb{R})$ such that $\alpha \wedge \beta$ is not zero. This is impossible since $\beta$ should also come from the cohomology of the curve $C$ and $H^3(C, \mathbb{R}) = 0$. $\square$

Proof of the theorem 8.6. Let $G$ be an infinite group which is the fundamental group of a closed 3-manifold $M$. We want to show that $G$ is not Kähler. First note that we can assume that $M$ is oriented since a non-orientable manifold has a 2-covering by an orientable one and since finite index subgroups of Kähler groups are Kähler.

If $M$ is not prime, then its fundamental group is the free product of the fundamental groups of the prime components. By the theorem 6.2, this implies that all prime components except one are simply connected (the theorem of Poincaré-Perelman shows that there are in fact no such components but we don’t really need it) and the last component has the fundamental group of $M$. So we can assume that $M$ is an oriented compact and prime 3-manifold.

$M$ cannot be homeomorphic to $S^1 \times S^2$ since this space has fundamental group $\mathbb{Z}$ which is not Kähler. So by 8.8 and 8.10, $M$ is irreducible, hence aspherical. Hence $M$ is a classifying space of its fundamental group and the cohomology algebra of $\pi_1(M)$ is the one of $M$. In particular it satisfies 3-dimensional oriented Poincaré duality.

If $b_1(M)$ is positive, then 8.12 gives the desired conclusion. Finally, if $b_1(M) = 0$, we know by proposition 8.11 that $M$ has a finite covering with positive first Betti number or $M$ is hyperbolic. In the first case, we can again apply 8.12; in the second case, the theorem 6.3 gives the conclusion. $\square$

Remark 8.5.

- Thurston elliptization conjecture (which is a theorem since Perelman’s work) states that a 3-manifold $M$ with finite fundamental group $\Gamma$ is a spherical 3-manifold $S^3/\Gamma$, where $\Gamma$ acts freely on $S^3$ by isometries, hence is a subgroup of $S^3$. This improves theorem 8.6.

- Lemma 8.12 could maybe be improved. Whether there exists or not a Kähler group $G$ with cohomological dimension (in reals or integers) 3 is still unknown. Note that the first part of the proof of this lemma shows that if $X$ is a closed Kähler manifold such that $\pi_1(X) = G$, then $X$ has 1-dimensional Albanese image. But then one needs oriented Poincaré duality to go on.

Kotschick asked me to work on this topic but it didn’t seem that we could conclude with his ideas, even in the case where the fibration from $X$ to its Albanese image is nice (i.e. without multiple fibers).
Chapter 9

Gromov-Green-Lazarsfeld theorem

In this chapter, we study the deficiency of the fundamental group of a compact Kähler manifold to show that it has 1-dimensional Albanese image, or at least fibers over a curve whose genus is well controlled.

9.1 Gromov-Green-Lazarsfeld theorem

The deficiency of a finitely presentable group $G$ is a useful algebraic invariant of $G$.

**Definition 9.1 (Deficiency).** Let $G$ be a finitely presentable group. The deficiency of a presentation of $G$ with $p$ generators and $q$ relations is $p-q$. The deficiency of $G$ denoted by $\text{def}(G)$ is the maximum of the deficiencies of its presentations.

**Lemma 9.1.** The first two Betti numbers of a group $G$ over any field $F$ satisfy $b_1(G; F) - b_2(G; F) \geq \text{def}(G)$.

**Proof.** Let $G = \langle a_1, \ldots, a_p | r_1, \ldots, r_q \rangle$ be a presentation of $G$ with deficiency $\text{def}(G)$. Then one can construct the Eilenberg-Mac Lane space $Y$ of $G$ by first considering the wedge product of $p$ circles then attaching $q$ cells of dimension 2 to kill the relations and finally attaching possibly infinitely many cells of higher dimensions. In particular $Y$ has 1 cell of dimension 0, $p$ cells of dimension 1 and $q$ cells of dimension 2. Its cellular chain complex looks as follows

$$
\ldots \xrightarrow{\partial_3} F^q \xrightarrow{\partial_2} F^p \xrightarrow{\partial_1} F \to 0.
$$

One has $b_i(Y; F) = b_i(G; F)$ by definition and

$$
b_1(Y; F) - b_2(Y; F) = \dim(H^1(Y)) - \dim(H^2(Y))
= \dim(\text{Ker}(\partial_1)) - \dim(\text{Im}(\partial_2)) - \dim(\text{Ker}(\partial_2)) + \dim(\text{Im}(\partial_3))
= p - q + \dim(\text{Im}(\partial_3)) - \dim(\text{Im}(\partial_1)).
$$

But since the 1-skeleton of $Y$ is a wedge of circles, $\partial_1$ is the zero map and we have

$$
b_1(Y; F) - b_2(Y; F) = \text{def}(G) + \dim(\text{Im}(\partial_3)) \geq \text{def}(G).
$$

\[\square\]

This gives a useful criterion to show that a Kähler manifold has 1-dimensional Albanese image:
Theorem 9.3.

- Let $X$ be a compact Kähler manifold. If $\pi_1(X)$ surjects onto a group $G$ whose deficiency is $\geq 2$ then $X$ admits a surjective holomorphic map with connected fibers $f : X \to C$ to a curve $C$ with $g(C) \geq \frac{1}{2}(\text{def}(G) + b_2(G; \mathbb{R}))$.
- Moreover if $\text{def}(\pi_1(X)) \geq 2$ then $X$ has 1-dimensional Albanese image.

Proof.

- Denote by $Y$ the Eilenberg-MacLane space of $G$ and consider some $\alpha \neq 0 \in H^1(Y; \mathbb{R})$. The cup product with $\alpha$ is a morphism $\tilde{\alpha}$ from $H^1(Y; \mathbb{R})$ to $H^2(Y; \mathbb{R})$ and $\dim(\text{Ker} \tilde{\alpha}) \geq b_1(Y; \mathbb{R}) - b_2(Y; \mathbb{R}) \geq 2$ by the above lemma and the assumption about def($G$). Thus $\alpha$ is contained in an isotropic subspace of dimension at least 2.

Denote by $\phi$ a surjective morphism from $\pi_1(X)$ to $G$ and by $B\phi$ the corresponding map from $B\pi_1(X)$ to $BG = Y$. Then $\phi^* = (B\phi)^* : H^1(Y; \mathbb{R}) \to H^1(B\pi_1(X); \mathbb{R})$ is injective. Composing with the tautological map of $X$ which is an isomorphism on $H^1$ we get an injective map from $H^1(Y; \mathbb{R})$ to $H^1(X; \mathbb{R})$. Denote the image of $H^1(Y; \mathbb{R})$ by $V \subset H^1(X; \mathbb{R})$.

Each $\alpha \in V$ is contained in an isotropic subspace of $H^1(X; \mathbb{R})$ of dimension at least 2. Hence by the Castelnuovo-de Franchis theorem, each $\alpha$ is in the image of the pullback $f^*$ of some fibration $f$ from $X$ to a curve $C_g$. The countability argument applied in the proof of theorem 8.5 shows that $V$ itself is contained in the image of the pullback of one of these maps $f$. For such a map $f$ one has necessarily

$$g := \frac{\dim(H^1(C_g); \mathbb{R})}{2} \geq \frac{\dim(V)}{2} = \frac{b_1(Y; \mathbb{R})}{2}.$$  

Since by the lemma $b_1(Y; \mathbb{R}) \geq \text{def}(G) + b_2(Y; \mathbb{R})$ this proves the first part of the theorem.

- If $\pi_1(X)$ has deficiency at least 2 then the first part of the proof shows that every $\alpha \in H^1(B\pi_1(X); \mathbb{R})$ is contained in an isotropic subspace of dim $\geq 2$. Since the tautological map of $X$ is an isomorphism on $H^1$ this is also true for $X$: every $\alpha \in H^1(X; \mathbb{R})$ is contained in an isotropic subspace of dim $\geq 2$ for the cup product. This implies that $X$ has a 1-dimensional Albanese image (see theorem 8.5).

\[ \square \]

9.2 A counterexample for the converse

The second part of the theorem has no converse. Indeed one can show that $\mathbb{Z}^2$, which is the fundamental group of an elliptic curve has deficiency 1. In order to avoid such an obvious example one can consider groups with first Betti number over $\mathbb{C}$ greater than 2. But we have the following theorem.

Theorem 9.3. For every $g > 1$, there exist compact Kähler manifolds whose Albanese images are curves of genus $g$ and whose fundamental groups have arbitrarily large negative deficiency.

We need three lemmas.

Lemma 9.4. If $G$ is a Kähler group with Albanese image a curve $C_g$, then so is $G \times H$ for every finite group $H$. 
9.2. A COUNTEREXAMPLE FOR THE CONVERSE

Proof. Since a finite group has no cohomology in degree $>0$ with complex coefficients (XXX), the cup-product

$$H^1(G \times H; \mathbb{C}) \times H^1(G \times H; \mathbb{C}) \to H^2(G \times H; \mathbb{C})$$

is the same as the cup product $H^1(G; \mathbb{C}) \times H^1(G; \mathbb{C}) \to H^2(G; \mathbb{C})$. By theorem 8.5, this proves that the Albanese image of $G \times H$ is also $C_g$. □

Lemma 9.5. There exist finite groups with arbitrarily large negative deficiency.

Proof. Consider the product $G_n := (\mathbb{Z}/2\mathbb{Z})^n$. It has a natural presentation

$$G_n = \langle a_1, \ldots, a_n | a_1^2, \ldots, a_n^2, (a_i a_j a_i^{-1} a_j)^{1 \leq i < j \leq n} \rangle.$$ (9.1)

The deficiency of this presentation is $n - (n + \binom{n}{2}) = -\binom{n}{2}$. We claim that this is the deficiency of the group.

Consider the infinite real projective space which is the Eilenberg-Mac Lane space of $\mathbb{Z}/2\mathbb{Z}$. Over the ground field $\mathbb{F}_2$ its homology has dimension 1 in any degree.

One can compute the homology of $G_n$ by the Künneth formula which is true over any field,

$$b_1(G_n; \mathbb{F}_2) = n b_1(\mathbb{Z}/2\mathbb{Z}; \mathbb{F}_2) = n;$$

$$b_2(G_n; \mathbb{F}_2) = n b_2(\mathbb{Z}/2\mathbb{Z}; \mathbb{F}_2) + \binom{n}{2} b_1(\mathbb{Z}/2\mathbb{Z}; \mathbb{F}_2)^2$$

$$= n + \binom{n}{2}.$$

By the lemma (9.1), we obtain $\text{def}(G_n) \leq -\binom{n}{2}$ which implies the equality. □

Lemma 9.6. If $G$, $H$ are two finitely presentable groups, then $\text{def}(G \times H)$ is at most $\text{def}(H) + \text{rk}(G)$, where $\text{rk}(G)$ is the minimal number of generators for $G$.

Proof. Consider a presentation $P$ of $G \times H$ with maximal deficiency. Write the $\text{rk}(G)$ generators of $G$ as words in the generators in $P$. Then a presentation $P'$ of $H$ is obtained by adding these $\text{rk}(G)$ words as relations in $P$. Thus we obtain $\text{def}(H) \geq \text{def}(G \times H) - \text{rk}(G)$ □

With these lemmas, we can prove the theorem 9.3.

Proof. Choose a Kähler group $\Gamma$ with Albanese image $C_g$ - for instance $\pi_1(C_g)$ itself. Then the groups $\Gamma \times G_n$ (as in (9.5)) also have $C_g$ as Albanese image by (9.4). And by lemmata (9.5) and (9.6) their deficiency is bounded above by $-\binom{n}{2} + \text{rk}(\Gamma)$ which tends to $-\infty$ when $n$ tends to $+\infty$. □

One could ask if this is a torsion-phenomenon, i.e. if the converse is true for torsion-free groups. The following construction shows that this is not the case.

Theorem 9.7. There exists an aspherical compact smooth projective surface $S$ which has the same Betti numbers over $\mathbb{C}$ than the projective plane and whose fundamental group $\Gamma$ is torsion-free. Such a surface is called a fake projective plane.

Remark 9.1. The construction of such a surface $S$ was first given by D. Mumford in [Mum79]. Some considerations on the Chern classes of $S$ prove that $S$ is aspherical.

Since $S$ is aspherical, lemma 9.1 implies that $\text{def}(\Gamma) \leq b_1(\mathbb{C}P^2, \mathbb{C}) - b_2(\mathbb{C}P^2, \mathbb{C}) = -1$. Then the same proof shows that the product of $\pi_1(C_g^n)$ and $\Gamma^n$ for arbitrarily large $n$ gives examples of torsion-free Kähler groups which satisfy the hypothesis of Theorem 9.3.
Chapter 10

Some inequalities in the non-fibered case

In the preceding chapter, we proved the Gromov-Green-Lazarsfeld theorem which can be interpreted as a sufficient condition on the fundamental group of a compact Kähler manifold \( X \) to decide if it fibers or not over a curve of genus at least 2. In this chapter, we get some restrictions on the cohomology of \( \pi_1(X) \) when \( X \) does not fiber.

We say that a compact Kähler manifold \( X \) (resp. a Kähler group \( G \)) does not fiber if \( X \) (resp. \( G \)) does not fiber over any curve \( C_g \) with \( g \geq 2 \). The main theorem is the following:

**Theorem 10.1.** Let \( G \) be a Kähler group that does not fiber, such that \( b_1(G) \geq 2 \). Then
\[
b_2(G) \geq \max(3b_1(G) - 7, 1).
\]

10.1 Inequalities in spaces of \( (p,q) \)-forms

In the following two theorems, \( X \) is a compact Kähler manifold that does not fiber.

**Theorem 10.2.** The cup-product \( H^{1,0}(X) \times H^{1,0}(X) \to H^{2,0}(X) \) has rank \( \geq 2h^{1,0}(X) - 3 \).

**Theorem 10.3.** The cup-product \( H^{1,0}(X) \times H^{0,1}(X) \to H^{1,1}(X) \) has rank \( \geq 2h^{1,0}(X) - 1 \).

**Proof of 10.2.** We have the natural map \( c : \Lambda^2_C H^{1,0}(X) \to H^{2,0}(X) \) and, by the Castelnuovo-de Franchis theorem, \( c(\alpha_1 \wedge \alpha_2) \neq 0 \) whenever \( \alpha_1 \wedge \alpha_2 \neq 0 \) since \( X \) does not fiber. Denote by \( K \) the kernel of the linear map \( c \) and by \( C \) the cone of decomposable elements in \( \Lambda^2_C H^{1,0}(X) \); we can rewrite our assumption as \( K \cap C = \{0\} \).

In the projectivization of \( \Lambda^2_C H^{1,0}(X) \), the subvarieties \( \mathbb{P}(K) \) and \( \mathbb{P}(C) \) have to be disjoint. This implies that
\[
dim \mathbb{P}(K) + \dim \mathbb{P}(C) \leq \dim \mathbb{P}(\Lambda^2_C H^{1,0}(X)) - 1.
\]

Notice that \( \mathbb{P}(C) \) can be identified as the Grassmannian of 2-planes in \( H^{1,0}(X) \), via the Plücker embedding (see [H09]). So \( \dim \mathbb{P}(C) = 2(h^{1,0}(X) - 2) \). This gives
\[
dim K \leq \dim \Lambda^2_C H^{1,0}(X) - 2h^{1,0}(X) + 3
\]

Hence \( \text{rk}(c) \geq 2h^{1,0}(X) - 3 \). \qed

**Proof of 10.3.** Fix \( \omega \neq 0 \in H^{1,0}(X) \) and suppose for contradiction that \( \xi \neq 0 \in H^{0,1}(X) \) is such that \( \omega \wedge \xi = 0 \). Since we know that \( \omega \wedge \bar{\omega} \neq 0 \), \( \xi \) is not a multiple of \( \bar{\omega} \) and so \( \xi \) is not a multiple of \( \omega \). In particular \( \omega \wedge \xi \neq 0 \) by the Castelnuovo-de Franchis theorem.

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Since $\omega \land \bar{\xi}$ is a holomorphic form, the proposition 3.3 implies that
\[(\omega \land \bar{\xi}) \land (\omega \land \xi) \neq 0,
\]
contradicting $\omega \land \xi = 0$.

This shows in particular that there is an induced map $\mathbb{P}(H^{1,0}(X)) \times \mathbb{P}(H^{0,1}(X)) \to \mathbb{P}(F)$ where $F$ denotes the image of the cup-product. The restrictions to each $\{x\} \times \mathbb{P}(H^{0,1}(X))$ or $\mathbb{P}(H^{1,0}(X)) \times \{y\}$ are embeddings since they come from injective linear maps. The following lemma concludes the proof.

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**Lemma 10.4.** Let $f$ be a holomorphic map from $\mathbb{C}P^k \times \mathbb{C}P^l$ to $\mathbb{C}P^m$ whose restrictions to components $\{x\} \times \mathbb{C}P^l$ or $\mathbb{C}P^k \times \{y\}$ are embeddings. Then $m \geq k + l$.

**Proof.** Let $\omega$ be a Kähler form on $\mathbb{C}P^m$. Denote by $x$ a point in $\mathbb{C}P^k$ and by $\iota_x$ the inclusion $\{x\} \times \mathbb{C}P^l \hookrightarrow \mathbb{C}P^k \times \mathbb{C}P^l$. By assumption $f \circ \iota_x$ is an embedding of $\mathbb{C}P^k$ in $\mathbb{C}P^m$, hence the integral $\int_{\{x\} \times \mathbb{C}P^l}((f^* \circ \iota_x^*)\omega)^k$ is positive. Moreover, it does not depend on the point $x$ since $\omega$ is closed. There is an analogous statement for points $y$ in $\mathbb{C}P^l$.

Hence we get the equality
\[
\int_{\mathbb{C}P^k \times \mathbb{C}P^l} (f^* \omega)^{k+l} = \int_{\{x\} \times \mathbb{C}P^l} (\iota_x^* \circ f^* \omega)^k \times \int_{\mathbb{C}P^k \times \{y\}} (\iota_y^* \circ f^* \omega)^l,
\]
which is strictly positive. In particular $\omega^{k+l}$ is non-zero, which implies $2m \geq k + l$.

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**10.2 Inequalities for group invariants**

**Theorem 10.5.** Let $X$ be a compact Kähler manifold that does not fiber. Then
\[b_2(X) \geq \max(3b_1(X) - 7, 1).
\]

More precisely the rank of the cup product $H^1(X, \mathbb{R}) \times H^1(X, \mathbb{R}) \to H^2(X, \mathbb{R})$ is at least \[\max(3b_1(X) - 7, 1)\] if $b_1(X) \geq 2$.

**Proof.** Note that the second statement implies the first except when $b_1(X) = 0$. But since the Kähler form is non-trivial in $H^2$, this case is clear and we just prove the second statement.

The rank of the cup-product $H^1(X, \mathbb{R}) \times H^1(X, \mathbb{R}) \to H^2(X, \mathbb{R})$ is obviously two times the rank involved in 10.2 plus the rank involved in 10.3. So the total rank is at least $2(2h^{1,0}(X) - 3) + (2h^{1,0}(X) - 1) = 3b_1(X) - 7$. Moreover it is at least 1 by 10.3.

**Remark 10.1.** Since the cup-product is the same for $X$ and for $\pi_1(X)$, the second statement of the theorem is also true when one considers the Betti numbers of $\pi_1(X)$; in particular we get theorem 10.1. This also leads to another statement about the deficiency of Kähler groups.

**Theorem 10.6.** Let $G$ be a Kähler group that does not fiber. Then $\text{def}(G) \leq 7 - 2b_1(G)$.

**Proof.** We know that $\text{def}(G) \leq b_1(G) - b_2(G)$. So the above remark gives the desired inequality.

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Bibliography


BIBLIOGRAPHY


