BANACH-COLMEZ SPACES

<u>Reminder from last time:</u>

E local field, residue field $\mathbb{F}_q$, unif. π $<$ $[E:\mathbb{Q}_p]<\infty$ / $E\simeq\mathbb{F}_q((\pi))$.

$S\in \mathrm{Perf}_{\mathbb{F}_q}=\{$cat. of perf'd spaces $/\mathbb{F}_q\}$

$\rightsquigarrow$ $X_{S,E}=Y_{S,E}/\varphi_S$, with $Y_{S,E}=\mathrm{Spa}\,W_{\mathcal{O}_E}(R^+)\setminus V(\pi[\varpi])$ if $S=\mathrm{Spa}(R,R^+)$, p.u. ϖ.

When $S=\mathrm{Spa}(C,C^\circ)$, C complete alg closed val. field $\supseteq\mathbb{F}_q$, classification of vb: the functor $\mathrm{Isoc}_E\to \mathrm{VB}(X_{C,E})$ is essentially surjective, ie any vb $\mathcal{E}$ on $X_{C,E}$ can be written $\mathcal{E}=\bigoplus_{i=1}^{r}\mathcal{O}_{X_{C,E}}(\lambda_i)$, $\lambda_i\in\mathbb{Q}$.

<u>Rk</u>: Classification of line bundles is already interesting: uses Lubin-Tate theory. If $S=\mathrm{Spa}(R,R^+)$, $S^\#=\mathrm{Spa}(R^\#,R^{\#+})$ untilt of S over E, have an identification

$$H^0(X_{C,E},\mathcal{O}_{X_{C,E}}(1))\simeq\varprojlim_{\times\pi} G(R^{\#+}),$$ G LT formal gp law for E over $\mathcal{O}_E$.

<u>What about general S?</u>

<u>Def</u>: Let $S\in\mathrm{Perf}_{\mathbb{F}_q}$.

* A <u>vector bundle</u> on $X_{S,E}$ is a finite loc free $\mathcal{O}_{X_{S,E}}$-module.

* A <u>flat coherent sheaf</u> on $X_{S,E}$ is an $\mathcal{O}_{X_{S,E}}$-module which can be written locally on $X_{S,E}$ as the cokernel of a fiberwise (on S) injective morphism of v.b.

<u>Ex</u>: Any vb is a flat coh sheaf. If $S^\#$ untilt of S, get $i_{S^\#}: S^\#\to X_{S,E}$, $i_{S^\#,*}\mathcal{O}_{S^\#}$ is also a flat coh sheaf.

<u>Def</u>: The <u>v-site</u> is the Grothendieck topology on $\mathrm{Perf}_{\mathbb{F}_q}$ for which a collection $\{f_i: S_i\to S\}_{i\in I}$ of morphisms is a covering if for each $U\subseteq S$ qc. open, there exists $J\subseteq I$ finite, $U_i\subseteq S_i$ qc open for each $i\in J$, s.t. $U=\bigcup_{i\in J}f_i(U_i)$.

② Rk: 1) Counter-example: $\{*\} \sqcup \mathbb{D}_K^* \to \mathbb{D}_K$ not a covering.

2) $f : T \to S$ morphism of analytic adic spaces. Then $|f| : |T| \to |S|$ is generalizing (use that set of generalizations of a pt is Spec(valuation ring))

In particular, if f surjective, $|f|$ quotient map.

Th/Prop: The v-site is subcanonical (in particular, $\mathcal{O}$, $\mathcal{O}^+$ are v-sheaves)

Th: The prestacks on $\mathrm{Perf}_{\mathbb{F}_q}$

$\mathrm{Bun} : S \longmapsto$ gpd of vector bundles on $X_{S,E}$

$\mathrm{Coh}^{fl} : S \mapsto$ gpd of flat coh sheaves on $X_{S,E}$

are v-stacks.

Rk: v-descent is useful for reducing questions about flat coherent sheaves on $X_{S,E}$ to the case $S = \mathrm{Spa}(C, C^+)$ geometric point. E.g., one can prove that for any $\mathcal{F} \in \mathrm{Coh}^{fl}(S)$, there exists a short exact sequence:

$$0 \to \mathcal{O}_{X_{S,E}}(n-1)^a \to \mathcal{O}_{X_{S,E}}(n)^b \to \mathcal{F} \to 0$$

for some $n \in \mathbb{Z}$, $a, b \in \mathbb{N}$, v-loc. on S.

<u>Cohomology of vector bundles and flat coherent sheaves</u>

Prop: Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{F} \in \mathrm{Coh}^{fl}(S)$. The functor $\mathrm{Perf}_S \ni T \mapsto R\Gamma(X_{T,E}, \mathcal{F}_{|X_{T,E}})$ is a v-sheaf of complexes, denoted $R\tau_* \mathcal{F}$. $(\tau : X_{S,E,v} \to S_v)$.

If all slopes of $\mathcal{F}$ (fiberwise on S) are ≥ 0, $R^i\tau_*\mathcal{F} = 0$ if $i \neq 0$. Write $BC(\mathcal{F}) = \tau_*\mathcal{F}$ in this case.

If ——— are < 0, $R^i\tau_*\mathcal{F} = 0$ if $i \neq 1$.

Write $BC(\mathcal{F}) = R^1\tau_*\mathcal{F}$ in this case.

Examples: a) $BC(\mathcal{O}) = \underline{E}$ $\quad (S \mapsto C^0(|S|, E))$.

b) $BC(\mathcal{O}(1)) = \mathrm{Spa}\, \mathbb{F}_q[[T^{1/p^\infty}]]$ (not analytic, but becomes so after base change to any $S \in \mathrm{Perf}_{\mathbb{F}_q}$)

c) $BC(i_{S^\#*}\mathcal{O}_{S^\#}) = \mathbb{A}^1_{S^\#}$.

X_S is really not a product $X \times S$!

d) Using exact seq: (after choosing untilt $S^\#$) ③

$$0 \to \mathcal{O}_{X_{S,E}}(-1) \to \mathcal{O}_{X_{S,E}} \to i_{S^\# *}\mathcal{O}_{S^\#} \to 0$$

get $BC(\mathcal{O}(-1)) = \mathbb{A}^1_{S^\#}/\underline{E}$.

Two striking facts:

1) (Structure of $\mathcal{P}ic$) Locally for the v-topology, any line bundle fiberwise (on S) ~~isomorphic to $\mathcal{O}$~~ semi-stable of degree d on $X_{S,E}$ is isomorphic to $\mathcal{O}_{X_{S,E}}(-d)$ $(d \in \mathbb{Z})$.

Therefore, $\mathcal{P}ic = \bigsqcup_{d\in\mathbb{Z}} \mathcal{P}ic^d \simeq \bigsqcup_{d\in\mathbb{Z}} [\mathrm{Spa}\,\mathbb{F}_q/\underline{E}^\times]$

Very different from Picard stack of a "classical" curve!

More generally, Bun_n contains $[\mathrm{Spa}\,\mathbb{F}_q/\underline{GL_n(E)}]$ as an open substack.

2) The functor $R\tau_*: \mathrm{Perf}(X_{S,E}) \longrightarrow D(S_v, \underline{E})$ is fully faithful.

In particular, $R\tau_*(R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X_{S,E}}}(\mathcal{E},\mathcal{O})) \simeq R\,\underline{\mathrm{Hom}}_{\underline{E}}(R\tau_*\mathcal{E}, \underline{E})$

(« Serre duality on the FF-curve)

[Statement is proved by reduction to old result of Breen computing self-ext of $\mathbb{G}_a$ on the perfect site of $\mathrm{Spec}(\mathbb{F}_q)$].

Diamonds:

<u>Def</u>: Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$. A <u>diamond</u> (over S) is a v-sheaf Y that can be written as:

$$Y = X/R$$

$X \in \mathrm{Perf}_S$, $R \subseteq X \times X$ eq. rel^n rep by perfectoid space, with pro-étale projection maps

Analogue in this setting of algebraic spaces.

Ex : 1) Any $X \in \mathrm{Perf}_S$ defines a diamond $X^\diamond = h_X$.

2) Let T ~~top.~~ compact Hausdorff space. On Perf_S, $\underline{T}$ is a diamond (even a perf'd space if T ~~has~~ profinite)

(trick: Stone-Čech compactification of T_{dis})

3) $\{\text{analytic adic spaces}/\mathbb{Z}_p\} \longrightarrow \{\text{diamonds}\}$

$X \longmapsto X^\diamond : S \mapsto \{\text{untilts of } S^\# \text{ over } X\}/\sim$

If X perfectoid, $X^\diamond = X^{\flat\diamond}$ (tilting equivalence)

$X_{S,E}^\diamond \simeq (\mathrm{Spa}\, E^\diamond \times_{\mathrm{Spa}\, \mathbb{F}_q} S^\diamond)/\varphi_S$

Prop : Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{F} \in \mathrm{Coh}^{bh}(S)$ with either only non-negative slopes or negative slopes. Then $BC(\mathcal{F})$ is a diamond.

Rk : Recall from last time that:

$\{\text{degree 1 effective Cartier divisors on } X_{S,E}\} \overset{1:1}{\longleftrightarrow} \{\text{untilts } S^\# \text{ of } S \text{ over } E\}/\varphi$

Hence:

moduli of deg 1 Cartier divisors $\nearrow$ $\mathrm{Div}^1_E \simeq \mathrm{Spa}(E)^\diamond/\varphi$.

Also know that $\mathrm{Pic}^1 = [\mathrm{Spa}\, \mathbb{F}_q / \underline{E}^\times]$. (cf. above)

Hence, Abel-Jacobi morphism looks like; after b.c. to $\overline{\mathbb{F}_q}$

$\mathrm{Div}^1_{E,\overline{\mathbb{F}_q}} \simeq \mathrm{Spa}(\breve{E})/\varphi \xrightarrow{AJ} \mathrm{Pic}^1_{\overline{\mathbb{F}_q}} = [\mathrm{Spa}\, \overline{\mathbb{F}_q}/\underline{E}^\times]$

Fibers: $BC(\mathcal{O}(1)) \setminus \{0\}$.

Fargues' reformulation of local CFT: a rank 1 line sheaf on $\mathrm{Div}^1_{E,\overline{\mathbb{F}_q}}$ comes via pullback along AJ from a line sheaf on $\mathrm{Pic}^1_{\overline{\mathbb{F}_q}}$. (!)