FOURIER TRANSFORM

$k = \mathbb{F}_q$ finite field of char. p, $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$ non-trivial character
V finite dim'l k-vs, $f : V \to \overline{\mathbb{Q}}_\ell$.
Fourier transform $F_\psi f : V^\vee \to \overline{\mathbb{Q}}_\ell$, $y \longmapsto \sum_{x \in V} f(x)\, \psi(\langle x, y\rangle)$

<u>Geometrization</u> (Deligne-Laumon):
S scheme (or stack) over k, $\mathcal{V} \to S$ (geometric) vector bundle, with dual $\mathcal{V}^\vee \to S$.
Let ψ as before, $\mathcal{L}_\psi \in D_{ét}(\mathbb{A}^1_S, \overline{\mathbb{Q}}_\ell)$ associated Artin-Schreier sheaf

$$\begin{tikzcd} & \mathcal{V}^\vee \times_S \mathcal{V} \arrow[r, "\alpha"] \arrow[dl, "\check{\pi}"'] \arrow[dr, "\pi"] & \mathbb{A}^1_S \\ \mathcal{V}^\vee & & \mathcal{V} \end{tikzcd}$$

$$\mathcal{F}_{\psi, \mathcal{V} \to \mathcal{V}^\vee} : D_{ét}(\mathcal{V}, \overline{\mathbb{Q}}_\ell) \to D_{ét}(\mathcal{V}^\vee, \overline{\mathbb{Q}}_\ell)$$
$$A \longmapsto \check{\pi}_!(\pi^* A \otimes \alpha^* \mathcal{L}_\psi)$$

(functors are always derived unless explicitly mentioned)

If $\mathcal{V} = V \otimes_k \mathcal{O}_S$, $A \in D_{ét}(\mathcal{V}, \overline{\mathbb{Q}}_\ell)$,
$$\mathrm{tr}(\mathcal{F}_\psi(A)) = F_\psi(\mathrm{tr}_A).$$

<u>Prop</u>: The formation of $\mathcal{F}_\psi$ commutes with base change in S. It is an equivalence of categories, commuting with Verdier duality ($\Rightarrow$ preserves perverse sheaves)
↳ More precisely, $\mathcal{F}_{\psi, \mathcal{V}^\vee \to \mathcal{V}^{\vee\vee}} \circ \mathcal{F}_{\psi, \mathcal{V} \to \mathcal{V}^\vee} \simeq \iota_*$ — can. iso $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\vee\vee}$

<u>Application</u>: X/k smooth projective geom. connected, $F = k(X)$.
Recall from <u>Lecture 1</u>: fix $\sigma : \mathrm{Gal}(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_\ell)$ ev. unramified & irreducible

Have constructed using Fourier analysis + induction
$$f'_\sigma : P_n(F) \backslash GL_n(\mathbb{A}_F) / GL_n(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell$$
cusp. Hecke eigenfunction and want to show that f'_σ is left $GL_n(F)$-inv.

Let $i \geq 0$. Def Coh_i^{fl} = alg. stack of flat (over the base) coherent sheaves of generic rank i on X.

Let $\mathcal{E}_i \subseteq Coh_i^{fl}$ open s.t. if $\mathcal{E}_i$ universal coh sheaf over $\mathcal{E}_i$, $\mathcal{V}_i = \underbrace{\underline{Hom}(\omega_X^{\otimes i}, \mathcal{E}_i)}_{T \mapsto Hom_{X_T}(\omega_{X_T}^{\otimes i},\, \mathcal{E}_i|_{X_T})}$ is a vector bundle.

By Serre duality, $\mathcal{V}_i^\vee = \underline{Ext}^1(\mathcal{E}_i, \omega_X^{\otimes i+1})$

$$\begin{tikzcd}
& \mathcal{V}_i^\vee \times \mathcal{V}_i \arrow[dl] \arrow[dr] \arrow[r] & Ext^1(\omega_X^{\otimes i}, \omega_X^{\otimes i+1}) = \mathbb{A}^1_{\mathcal{E}_i} \\
\mathcal{V}_i^\vee & & \mathcal{V}_i
\end{tikzcd}$$

$\rightsquigarrow \mathcal{F}_{\psi,i} := \mathcal{F}_{\psi, \mathcal{V}_i \to \mathcal{V}_i^\vee} : D_{et}(\mathcal{V}_i, \overline{\mathbb{Q}}_\ell) \to D_{et}(\mathcal{V}_i^\vee, \overline{\mathbb{Q}}_\ell)$

Set $\mathcal{V}_i^\circ \subseteq \mathcal{V}_i = \{(\mathcal{F}, s),\ s \text{ injective}\}$, $\mathcal{V}_i^{\vee,\circ} = \{0 \to \omega_X^{\otimes i+1} \to \mathcal{F}' \to \mathcal{F} \to 0\}$, $\mathcal{F}' \in \mathcal{E}_{i+1}$

Note $\mathcal{V}_i^{\vee,\circ} \underset{\gamma_i}{\simeq} \mathcal{V}_{i+1}^\circ$

Obtain:

$$\begin{tikzcd}
\cdots & \mathcal{V}_2^\vee \arrow[dr] & & \mathcal{V}_2 \supseteq \mathcal{V}_2^\circ \simeq \mathcal{V}_1^{\vee,\circ} \subseteq \mathcal{V}_1^\vee \arrow[dl] \arrow[dr] & & \mathcal{V}_1 \supseteq \mathcal{V}_1^\circ \arrow[dl] \arrow[dr, "f"'] & \\
& & \mathcal{E}_2 \subseteq Coh_2^{fl} & & \mathcal{E}_1 \subseteq Coh_1^{fl} & & Coh_0^{fl}
\end{tikzcd}$$

$(\omega_X \to \mathcal{F}) \mapsto \mathcal{F}/\omega_X$

and functors $\mathcal{F}^\circ_{\psi,i} : D_{et}(\mathcal{V}_i^\circ, \overline{\mathbb{Q}}_\ell) \to D_{et}(\mathcal{V}_{i+1}^\circ, \overline{\mathbb{Q}}_\ell)$

$A \longmapsto \gamma_{i,*} \circ j_i^{\vee *} \circ \mathcal{F}_{\psi,i} \circ j_{i,!} A$

Composing (n-1)-times, get $A_{n,\psi} : D_{et}(\mathcal{V}_1^\circ, \overline{\mathbb{Q}}_\ell) \to D_{et}(\mathcal{V}_n^\circ, \overline{\mathbb{Q}}_\ell)$

Laumon (geometrization of Casselman-Shalika): construct $\mathcal{L}_{\mathbb{L}} \in D_{et}(Coh_0^{fl}, \overline{\mathbb{Q}}_\ell)$, $\mathbb{L}$ rk n local system on X $\leftrightarrow \sigma$

s.t. pull back to $X^{(d)}$ is $\mathbb{L}^{(d)}$ for any $d \geq 0$.

Claim (Frenkel-Kazhdan-Gaitsgory-Vilonen):

The restriction $Aut'_{\mathbb{L}} \in D_{et}(Bun_n, \overline{\mathbb{Q}}_\ell)$ of $A_{n,\psi}(f^*\mathcal{L}_{\mathbb{L}})$ to "$Bun_n \subseteq \mathcal{V}_n^\circ$" is s.t.

$$t_{Aut'_{\mathbb{L}}} = f_\sigma.$$

Rk: Restriction to $\mathcal{E}_i$ does not matter using action of Hecke operators.

Hard work: descent (uses perversity).

Let E local field, residue field $\mathbb{F}_q$, unif π.

Last time: defined Coh^{fl} ~~moduli~~ v-stack of "flat" coh sheaves on the FF-curve for E.

For $i \geq 0$, define $\mathcal{E}_i \underset{\text{open}}{\subseteq} \mathrm{Coh}^{fl}$ formed by those $\mathcal{F} \in \mathrm{Coh}^{fl}(S)$ s.t. $\forall s \to S$ geom. pt, $\mathcal{F}_s$ has generic rank i and ~~either~~ $\{ \mathcal{F}_s$ has HN-slopes ≥ 0 ~~$\mathcal{F}_s$ trivial~~

(Cheat: actual def slightly more restrictive)

If $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{F} \in \mathcal{E}_i(S)$,

$$R\tau_* \mathcal{F} = \tau_* \mathcal{F} = BC(\mathcal{F})$$ v-sheaf (even $\diamond$) of $\underline{E}$-vs. on Perf_S.

$$\{\text{v-sheaves of } \underline{E}\text{-vs}\} \subset \{\text{Picard stacks in } \underline{E}\text{-vs}\} \simeq D^{[-1,0]}(S_v, \underline{E})$$

($\sim$ Picard stacks with $\underline{E}$-action)

$$\mathcal{G} \longmapsto \mathcal{G}^b \quad \left(\begin{array}{l} \mathcal{H}^0(\mathcal{G}^b) = \pi_0(\mathcal{G}) \\ \mathcal{H}^1(\mathcal{G}^b) = \underline{\mathrm{Aut}}_{\mathcal{G}}(\text{unit}) \end{array} \right)$$

$$\mathbb{V}(K) \longleftarrow K$$

Given $\mathcal{G}$, get $\mathcal{G}^\vee := \mathcal{H}om(\mathcal{G}, [S/\underline{E}])$

$$((\mathcal{G}^\vee)^b = \tau_{\leq 0} R\mathcal{H}om_{S_v, \underline{E}}(\mathcal{G}^b, \underline{E}[1]).$$

Prop: If $\mathcal{F} \in \mathcal{E}_i(S)$, $S \in \mathrm{Perf}_{\mathbb{F}_q}$, natural map $BC(\mathcal{F})^\vee \to (BC(\mathcal{F}^\vee))^\vee$ is an isom.

If $\lambda \in \mathbb{Q}_{>0}$, $BC(\mathcal{O}(\lambda))^\vee \simeq BC(\mathcal{O}(-\lambda))$.

$$\underline{E}^\vee \simeq [S/\underline{E}].$$

Follows from the results of the last talk. (in particular, "Serre duality on the FF-curve")

<u>ℓ-adic sheaves</u>: Fix Λ torsion ring, killed by integer prime to p.

Scholze: For each (small) v-stack Y, define Λ-linear ~~stable~~ ~~∞~~ triangulated ~~cat~~ stable ∞-cat $D_{\text{ét}}(Y, \Lambda) \subset D(Y_v, \Lambda)$ + 6 functors satisfying usual properties

The functors $f_!$, $f^!$ are defined under some conditions (ensuring "finite coh. dimension").

necessary to have right adjoint of $f_!$.

<u>Rk</u>: Let $Y' \to Y$ morphism of v-stacks. Define a functor
$$\mathrm{Spa}(R,R^+) \mapsto Y'(R,R^\circ) \times_{Y(R,R^\circ)} Y(R,R^+).$$ Extends to a v-stack $Y'^{/Y} \to Y$ s.t. $Y' \to Y$ factors through $Y'^{/Y} \to Y$. "Canonical compactification" of $Y' \to Y$. If $Y = *$, $Y' = \mathrm{Spa}(R,R^+)$, $Y'^{/*} = \mathrm{Spa}(R, \widetilde{\mathbb{F}_p + R^{\circ\circ}})$.

$[\mathrm{Hom}((R, \widetilde{\mathbb{F}_p + R^{\circ\circ}}), (S,S^+)) = \mathrm{Hom}(R,S) = \mathrm{Hom}((R,R^+),(S,S^\circ)).]$

Can be used to define $f_!$, for (certain) f qc: give f = proper o open

Construction of $D_{et}(Y,\Lambda)$ is done using v-descent ($Y \mapsto D_{et}(Y,\Lambda)$ hyper-v-sheaf).

Fix $\psi : E \to \Lambda^*$ non-trivial character.

Let $i \geq 1$. Let $\mathcal{E}_i$ universal coh sheaf on $\mathcal{C}_1$, $\mathcal{V}_i = BC(\mathcal{E}_i)$.

$$\mathcal{V}_i^\vee \times \mathcal{V}_i \to [\mathcal{E}_i / \underline{E}] \qquad \mathcal{L}_\psi \in D_{et}([\cdot/\underline{E}], \Lambda).$$

$$\mathcal{V}_1^\vee \swarrow \qquad \searrow \mathcal{V}_1 \quad \rightsquigarrow \mathcal{F}_{\psi,i} := \mathcal{F}_{\psi, \mathcal{V}_i \to \mathcal{V}_i^\vee} : D_{et}(\mathcal{V}_1, \Lambda) \to D_{et}(\mathcal{V}_1^\vee, \Lambda).$$

<u>Prop</u>: There is a natural isom
$$\underbrace{\mathcal{F}_{\psi, \mathcal{V}_1^\vee \to \mathcal{V}_1^{\vee\vee}}}_{\mathcal{F}^\vee_{\psi,i}} \circ \mathcal{F}_{\psi,i} \simeq a_* \qquad a = \text{can isom } \mathcal{V}_i \simeq (\mathcal{V}_1^\vee)^\vee.$$

Hence $\mathcal{F}_{\psi,i}$ equivalence, & commuting with Verdier duality.

<u>Rk</u>: Compared to the classical situation, have replaced $\mathbb{A}^1$ by $[\cdot/\underline{E}]$, i.e. $\underline{\mathrm{Ext}}^1(\mathcal{O}_X, \omega_X)$ by the <u>stack</u> of extensions of $\mathcal{O}$ by $\mathcal{O}$. (reminiscent of what happens in local CFT: Fargues crucially needs Picard <u>stack</u>, not Picard "scheme").

<u>Ex</u>: $\mathrm{Spa}(C, C^\circ) \to \mathcal{C}_1 \leftrightarrow$ skyscraper sheaf at untilt $C^\#$ of C.

Then $\mathcal{F}_\psi : D_{et}(\mathbb{A}^1_{C^\#}, \Lambda) \to D_{et}(\mathbb{A}^1_{C^\#}, \Lambda)$
kernel defined by descending ψ along the $\underline{E}$-torsor
$$\widetilde{G} \to \mathbb{A}^1_{C^\#}, \quad G \text{ LT gp } / \mathcal{O}_E.$$