EXAMPLES

Let E local field. Let $n \geq 1$.
For each $r \geq 1$ s.t $n = rm$, let G_r be the algebraic gp
$G_r = GL_r(D)$ ⟵ division alg center E dim m^2.
If $r = n$, $G_r = GL_n$.

<u>Local Langlands</u> conjecture, special case: Fix $l \neq p$.

For each n, r as above, there exists an injective map

$$\left\{ \begin{matrix} \text{continuous irreducible} \\ \text{reps } W_E \to GL_n(\overline{\mathbb{Q}}_l) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{cuspidal reps of} \\ G_r(E) \text{ over } \overline{\mathbb{Q}}_l \end{matrix} \right\}$$
$$\sigma \longmapsto \pi_r(\sigma)$$

s.t.:
- if $n = r = 1$, local CFT.
- if $r = n$, the ε-factors of σ and $\pi_n(\sigma)$ coincide. Then, bijective!
- Character relation: $\chi_{\pi_r(\sigma)} = (-1)^{n-r} \chi_{\pi_n(\sigma)}$ on "elliptic reg. locus".

Known by work of Harris-Taylor, Henniart, Jacquet-Langlands, Deligne-Kazhdan-Vignéras.

Uniquely specified by these conditions + other natural properties.

Fargues' geometric reformulation:
Let b (isom class of) rank n <u>isoclinic</u> isocrystal for E over $\overline{\mathbb{F}}_q$.
unique slope $\lambda = \frac{d}{n}$. Let $r = d \wedge n$.
Then $\mathrm{Aut}(b) \simeq GL_r(D) = G_r$.
b isoclinic $\longleftrightarrow$ $\mathcal{E}_b$ semi-stable vector bundle.
Moreover (cf. Lecture 4):
$$\underset{\substack{\text{open in} \\ \mathrm{Bun}_n}}{\mathrm{Bun}_n^{ss}} = \bigsqcup_{b \text{ isoclinic}} [\cdot / \underline{G_r(E)}]. \quad \text{Write } i_b : [\cdot/G_r(E)] \hookrightarrow \mathrm{Bun}_n$$

(Very imprecise special case)

Fargues' conjecture: For any $\sigma : W_E \to GL_n(\overline{\mathbb{Q}}_l)$ cont. <u>irreducible</u>,
$\exists$ <u>Hecke</u> eigensheaf $\mathrm{Aut}_\sigma \in D_{et}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_l)$
st $i_b^* \mathrm{Aut}_\sigma \longleftrightarrow \pi_r(\sigma)[\circ]$

(strange) analogue of Hecke eigenvector condition in function field setting.

Note: We used implicitly the fact that:

$$D_{\acute{e}t}\left(\left[\cdot/GL_n(E)\right], \overline{\mathbb{Q}}_\ell\right) \simeq D\left(\mathrm{Rep}^{\infty}_{\overline{\mathbb{Q}}_\ell} GL_n(E)\right)$$

(true for any loc. profinite group)

Rk: LLC can be formulated for any reductive gp G/E, and Fargues' conjecture as well. For general G, it gives a beautiful geometric interpretation of the structure of L-packets.

Question: Can one prove the above conjecture geometrically, without using LLC?

Hope: Recall $\sigma \longleftrightarrow$ "local system" $\mathbb{L}$ on Div^1_E.
irreducible, rank n.

From now on, switch to coeff Λ torsion prime to p.

1) To $\mathbb{L}$, should be able to attach

$$\mathcal{L}_{\mathbb{L}} \in D_{\acute{e}t}(\mathrm{Coh}_0, \Lambda) \text{ s.t. pullback to}$$

Div^d_E is $\Lambda^{(d)}$ for all $d \geq 0$ (imitating Laumon's construction).

2) The restriction of $A_{n,\psi}(\mathcal{L}_{\mathbb{L}})$ to Bun'_n should descend along $\mathrm{Bun}'_n \to \mathrm{Bun}_n$ to the desired Hecke eigensheaf.

[$A_{n,\psi}: D_{\acute{e}t}(\mathrm{Coh}_0, \Lambda) \to D_{\acute{e}t}(\mathcal{V}_n^{\circ}, \Lambda)$ as defined in Lecture 5]

To provide some evidence, will consider 2 examples.

1) The slope 0 case

Let $i \geq 1$. Let us pullback the diagram

$$\begin{tikzcd}
 & \mathcal{V}_i^{\vee} \times \mathcal{V}_i \arrow[r] \arrow[dl] \arrow[dr] & {[\cdot/\underline{E}]} \\
\mathcal{V}_i^{\vee,0} \subseteq \mathcal{V}_i^{\vee} \arrow[dr] & & \mathcal{V}_i \supseteq \mathcal{V}_i^{0} \arrow[dl] \\
 & \mathcal{E}_i &
\end{tikzcd}$$

along the map $\mathrm{Bun}_i^{ss,0} \hookrightarrow \mathcal{E}_i$.

What we get can be explicitly described:

③

$$\begin{tikzcd}[column sep=small]
 & {[\underline{V_i}/\underline{P(V_i)}]} \arrow[r, "\alpha"] \arrow[dl] \arrow[dr] & {[\cdot/\underline{E}]} \\
{[\cdot/P(V_i)] = [\cdot/\underline{P(V_i)}]} \arrow[dr] & & {[\underline{V_i}/\underline{GL(V_i)}] \supseteq \left[\frac{(\underline{V_i}\setminus 0)}{\underline{GL(V_i)}}\right] \simeq [\cdot/\underline{P(V_{i-1})}]} \arrow[dl] \\
 & {[\cdot/\underline{GL(V_i)}]} &
\end{tikzcd}$$

where $\forall j$, $V_j = E^j$, $P(V_j) = GL(V_j) \ltimes V_j^\vee$
$= \text{mirabolic subgp in } GL(V_{j+1})$.

Claim: The functor $D_{et}([\cdot/\underline{P(V_{i-1})}], \Lambda) \longrightarrow D_{et}([\cdot/\underline{P(V_i)}], \Lambda)$
induced by $\mathcal{F}^\circ_{i,\psi}$

$$D(\mathrm{Rep}^\infty_\Lambda P(V_{i-1})) \longrightarrow D(\mathrm{Rep}^\infty_\Lambda P(V), \Lambda)$$

is the functor Φ^+_i (Bernstein–Zelevinsky),
sending a smooth rep π of $P(V_{i-1})$ to the compact-induction from $P(V_{i-1}).V_i$ to $P(V_i)$ of the rep given by π with V_i acting via ψ.

$\Uparrow$

Prop: Let V finite dim'l E-vs, fix Haar measure $d\check{v}$ on $\check{V}$.
Let $\mathcal{G} = \underline{V}$, $\mathcal{G}^\vee = [\cdot/\underline{V}^\vee]$. The functor
$$\mathcal{F}_{\psi, \mathcal{G}\to\mathcal{G}^\vee} : D_{et}(\mathcal{G}, \Lambda) \to D_{et}(\mathcal{G}^\vee, \Lambda)$$
is induced, via the identifications:
$$D_{et}(\mathcal{G}, \Lambda) \simeq D((C^\infty_c(\Lambda), \times)\text{-Mod}^{sm}),\quad D_{et}(\mathcal{G}^\vee, \Lambda) \simeq D(\mathrm{Rep}^\infty_\Lambda V^\vee) \simeq D((C^\infty_c(V^\vee\Lambda), *)\text{-Mod}^{sm})$$
by the isomorphism
$$(C^\infty_c(V^\vee, \Lambda), *) \simeq (C^\infty_c(V, \Lambda), \times)$$
$$f \mapsto \left(\hat{f} : v \mapsto \int_{\check{V}} f(\check{v})\psi(\check{v}(v))\, d\check{v}\right).$$

Moreover, expect that $f^*(\mathcal{L}_{\underline{1}})|_{\mathcal{V}^\circ_1 \times_{\mathcal{C}_0} \mathrm{Bun}_0^{ss,\circ} = *} \simeq \Lambda$.

Hence, as a representation of the mirabolic subgp $\subseteq GL_n(E)$, $i^*_{\underline{1}}(\mathrm{Aut}_\sigma)$ should be iso to $\Phi^+_{n-1} \circ \dots \circ \Phi^+_1$ (trivial).

$\rightsquigarrow$ "Kirillov model of a supercuspidal representation"

2) The <u>case</u> of $\mathcal{O}(\frac{1}{i})$.

Let us base-change the diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{V}_i^\vee \times \mathcal{V}_i \simeq [\cdot/\underline{E}] \arrow[dl] \arrow[dr] & \\ \mathcal{V}_1^{\vee\circ} \subseteq \mathcal{V}_i^\vee \arrow[dr] & & \mathcal{V}_i \supseteq \mathcal{V}_1^\circ \arrow[dl] \\ & \mathcal{C}_i & \end{tikzcd}$$

along $\mathrm{Bun}_i^{ss,1} \longrightarrow \mathcal{C}_i$.

Can again be made explicit:

$$\begin{tikzcd}[column sep=small] & & \cdots \arrow[dl] \arrow[dr] & & \\ \left[\frac{BC(\mathcal{O}(-\frac{1}{i}))\setminus 0}{D_i^\times}\right] \subseteq & \left[BC(\mathcal{O}(-\frac{1}{i}))/\underline{D}_i^\times\right] \arrow[dr] & & \left[BC(\mathcal{O}(\frac{1}{i}))/\underline{D}_i^\times\right] \arrow[dl] & \supseteq \left[\frac{BC(\mathcal{O}(\frac{1}{i}))\setminus 0}{D_i^\times}\right] \\ \rotatebox{90}{$\cong$} & & \left[\cdot/\underline{D}_i^\times\right] & & \\ \left[\frac{BC(\mathcal{O}(\frac{1}{i+1}))\setminus 0}{D_{i+1}^\times}\right] & & & & \end{tikzcd}$$

Hence, the functor induced by $\mathcal{F}_{i,\psi}^\circ$ is a functor

$$\mathcal{F}_{i,\psi}^\circ : D_{\acute{e}t}\left(\frac{BC(\mathcal{O}(\frac{1}{i}))\setminus 0}{D_i^\times}, \Lambda\right) \longrightarrow D_{\acute{e}t}\left(\frac{BC(\mathcal{O}(\frac{1}{i+1}))\setminus 0}{D_{i+1}^\times}, \Lambda\right)$$

Expect that $f^*(\mathcal{L}_{\mathbb{L}})|_{\mathcal{V}_1^\circ \times_{\mathcal{C}_1} \mathrm{Bun}_1^{ss,1}} = \left[\frac{BC(\mathcal{O}(1))\setminus 0}{E^\times}\right]$

is $\mathbb{L}$, via $\quad \mathrm{Div}_E^1 = \left[\frac{BC(\mathcal{O}(1))\setminus 0}{\underline{E}^\times}\right]$

Hence, predict that

$$\mathcal{F}_{n-1,\psi}^\circ \circ \cdots \circ \mathcal{F}_{1,\psi}^\circ (\mathbb{L}) \in D_{\acute{e}t}\left(\frac{BC(\mathcal{O}(\frac{1}{n}))\setminus 0}{D_n^\times}, \Lambda\right)$$

should come via pull back along $\left(BC(\mathcal{O}(\frac{1}{n}))\setminus 0\right)/D_n^\times \longrightarrow \cdot/D_n^\times$ from $\pi_1(\sigma)$. ⚠ No idea how to prove that!

<u>But</u>, <u>Rk</u>: $D_n^\times$ compact mod center, hence $\pi_1(\sigma)$ is finite dim'l.

Its dimension has been computed by Deligne/Carayol:

$n=2$ for <u>simplicity</u>

$$\mathrm{rk}_\Lambda\, \pi_1(\sigma) = \begin{cases} 2\, q^{sw(\sigma)/2} & \text{if } sw(\sigma) \text{ even} \\ (q+1)\, q^{\frac{sw(\sigma)}{2}-\frac{1}{2}} & \text{if } sw(\sigma) \text{ odd.} \end{cases}$$

Can we at least check that $\mathcal{F}_{1,\psi}^\circ(\mathbb{L})$ has the correct rk, when $\mathbb{L}$ has rk 2?

$\rightsquigarrow$ Yes, using the Grothendieck-Ogg-Shafarevich formula. (5)

Fix $\bar{x} : \mathrm{Spd}(C) \longrightarrow \mathcal{G}^{\nu,0}$ Let $\mathcal{G} = BC(\mathcal{O}(1))$.

(corresponding to a non-zero section of $\mathcal{O}(\frac{1}{2})$ over X_C)

Then $\bar{x}^* \mathcal{F}^{\circ}_{1,\psi}(\mathbb{L}) \simeq R\Gamma_c(\mathbb{D}^*_C, \tilde{\mathbb{L}}_C \otimes \mathcal{L}_{\psi,C})$

with $\tilde{\mathbb{L}}_C$ = pullback of $\mathbb{L}$ along $\mathbb{D}^*_C = BC(\mathcal{O}(1))\setminus\{0\} \times \mathrm{Spa}\, C$
local system.

$\mathcal{L}_{\psi,C}$ = pullback of $\mathcal{L}_\psi$.

$$\mathbb{D}^*_C \to BC(\mathcal{O}(1))\setminus\{0\} \to \mathrm{Div}^1.$$

Huber: The theory of Swan conductors exists for representations of the Galois group of any henselian valued field K s.t. $\Gamma_K = \mathbb{Z} \times$ (divisible gp).

e.g. K = discretely valued complete field ($\Gamma_K = \mathbb{Z}$)

or K = henselization of residue field at a rank 2 pt of a smooth analytic adic space of dim 1 / C alg closed complete.

$(\Gamma_K = \mathbb{Z} \times \Gamma_C)$

Γ_C divisible as C alg. closed.

$\rightsquigarrow$ GOS-formula:

Th (Huber) Let A Λ-local system on $\mathbb{D}^*_C$.

Assume $H^*_c(\mathbb{D}^*_C, A)$ finite, and let

$$\chi_c(\mathbb{D}^*_C, A) = \sum_{i=0}^{\infty} (-1)^i \ell_\Lambda(H^i_c(\mathbb{D}^*_C, A)).$$

Then: $$\chi_c(\mathbb{D}^*_C(A)) = -\,sw_{x_{<r}}(A) - sw_{x_{>r'}}(A)$$

for any r close to 0, r' close to 1.

Hence, need to compute the conductors close to the origin & boundary of $\tilde{\mathbb{L}}_C \otimes \mathcal{L}_{\psi,C}$

This is a fun computation, involving the inverse Herbrand function of E_∞ / E (recall that $BC(\mathcal{O}(1))\setminus\{0\} \simeq \mathrm{Spa}\, E_\infty^{\circ}$).

& Fontaine-Wintenberger field of norms theory.

$\rightsquigarrow$ recover Laumon's formulas. $\square$