

BMS filtrations on THH and its variants

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Let X be a CW-complex. To obtain the Atiyah-Hirzebruch spectral sequence computing the topological K -theory of X :

$$E_2^{i,j} = H^{i-j}(X, \mathbf{Z}(-j)) = \pi_{j-i}[X, H\mathbf{Z}(-j)] \implies K_{-i-j}^{\text{top}}(X) = \pi_{-i-j}[X, KU],$$

one can, instead of using the skeletal filtration on X , use the double speed *Postnikov filtration* on the K -theory spectrum KU (recall that the n -th Postnikov section $\tau_{<n}T$ of a spectrum T is obtained by killing all homotopy groups of T above dimension n by attaching cells ; the n -th piece of the Postnikov filtration is the homotopy fiber of the map $T \rightarrow \tau_{<n}T$, which is the n -connective cover of T).

The goal of the talk is, following [2], to construct a similar – but more involved – filtration, the *BMS filtration* (called the *motivic filtration* by the authors of [2]), on THH, TC^- , TP and TC over *quasi-syntomic rings* over a characteristic p perfect ring. If A is such a ring, $n \in \mathbf{Z}$, and

$$\mathbf{Z}_p(n)(A) := \text{gr}^n \text{TC}(A)[-2n]$$

is the (shifted) n -th graded piece of $\text{TC}(A)$ for the BMS filtration, one has a spectral sequence :

$$E_2^{i,j} = \pi_{j-i}(\mathbf{Z}_p(-j)(A)) \implies \pi_{-i-j} \text{TC}(A),$$

resembling the above Atiyah-Hirzebruch spectral sequence or the spectral sequence deduced from the filtration of algebraic K -theory by motivic cohomology. For the comparison with classical p -adic cohomology theories (as crystalline cohomology), the case of (quasi-)smooth rings over a perfect ring is probably the most interesting and doing the construction for general quasi-syntomic rings may seem like unnecessary generality ; it is actually a crucial feature of the argument.

The talk has two parts. We first introduce quasi-syntomic rings and state some properties of the quasi-syntomic site. Then we explain how to construct the filtrations, by combining some explicit computations with a descent argument. It follows very closely [2, §4, §6, §7].

1. THE QUASI-SYNTOMIC SITE

1.1. Quasi-syntomic rings. In the following, we will restrict to characteristic p , p being a fixed prime, though, as explained in [2], all the constructions extend to the mixed characteristic case.

Definition 1.1. Let A be a commutative ring, $M \in D(A)$ (the derived category of A -modules). If $a, b \in \mathbf{Z} \cup \{\pm\infty\}$, one says that M has *Tor-amplitude in $[a, b]$* if for every A -module N , $N \otimes^{\mathbf{L}} M \in D^{[a,b]}(A)$. One says that M is *flat* if it has Tor-amplitude in $[0, 0]$ (by definition, this means that M is concentrated in degree 0 and flat in the usual sense).

Definition 1.2. Let A be an \mathbf{F}_p -algebra.

(1) The \mathbf{F}_p -algebra A is *quasi-syntomic* if the cotangent complex $L_{A/\mathbf{Z}} \in D(A)$ has Tor-amplitude in $[-1, 0]$. Let QSyn denote the category of all quasi-syntomic \mathbf{F}_p -algebras.

Let $A \rightarrow B$ be a morphism of \mathbf{F}_p -algebras.

(2) One says that $A \rightarrow B$ is a *quasi-smooth map* (resp. a *quasi-smooth cover*) if it is flat (resp. faithfully flat) and if $L_{B/A} \in D(B)$ is flat.

(3) One says that $A \rightarrow B$ is a *quasi-syntomic map* (resp. a *quasi-syntomic cover*) if it is flat (resp. faithfully flat) and if $L_{B/A} \in D(B)$ has Tor-amplitude in $[-1, 0]$.

We endow the category QSyn^{op} with the topology defined by quasi-syntomic covers.

Remark 1.3. A theorem of Avramov says that a Noetherian ring A is a local complete intersection ring if and only if $L_{A/\mathbf{Z}}$ has Tor-amplitude in $[-1, 0]$. Therefore, the above definition extends the classical definition of a syntomic ring to the non-Noetherian setting.

Example 1.4. Any perfect \mathbf{F}_p -algebra R is a (usually non-Noetherian !) quasi-syntomic ring : the cotangent complex $L_{R/\mathbf{Z}}$ has Tor-amplitude in $[-1, -1]$ and is isomorphic to $R[1]$. Indeed, the composition $\mathbf{Z} \rightarrow \mathbf{F}_p \rightarrow R$ gives rise to a triangle

$$R \otimes_{\mathbf{F}_p}^{\mathbb{L}} L_{\mathbf{F}_p/\mathbf{Z}} \rightarrow L_{R/\mathbf{Z}} \rightarrow L_{R/\mathbf{F}_p}.$$

Because $\mathbf{F}_p = \mathbf{Z}/p$, $L_{\mathbf{F}_p/\mathbf{Z}}$ is simply $p\mathbf{Z}/p^2\mathbf{Z}[1] \simeq \mathbf{F}_p[1]$. Hence it suffices to show that

$$L_{R/\mathbf{F}_p} = 0.$$

To see this, choose a simplicial resolution R_\bullet of R by polynomial \mathbf{F}_p -algebras. The assumption that R is perfect implies that the Frobenius map Φ_{R_\bullet} induces an isomorphism $L_{R/\mathbf{F}_p} \simeq L_{\Phi_* R/\mathbf{F}_p}$. But for any k , if one identifies R_k with a polynomial algebra $\mathbf{F}_p[X_1, X_2, \dots]$, Φ_{R_k} sends X_i to X_i^p , thus is the zero map on differentials. This proves the claim.

Lemma 1.5. *The category QSyn^{op} with the quasi-syntomic topology forms a site.*

The only non trivial thing to check is that pull-backs of covers exist ; this is an easy exercise.

1.2. Quasi-regular semi-perfect rings. Once again, we restrict to the characteristic p setting.

Definition 1.6. An \mathbf{F}_p -algebra S is *quasi-regular semi-perfect* if $S \in \text{QSyn}$ and if there exists a surjective morphism $R \rightarrow S$, with R perfect (in particular, S is semi-perfect, i.e. Frobenius is surjective). We denote by QRSPerf the category of quasi-regular semi-perfect \mathbf{F}_p -algebras and endow $\text{QRSPerf}^{\text{op}}$ with the topology defined by quasi-syntomic covers.

Remarks 1.7. (a) If $S \in \text{QRSPerf}$, $L_{S/\mathbf{Z}}$ actually has Tor-amplitude in degrees $[-1, -1]$. Indeed, as S is semi-perfect, $L_{S/\mathbf{Z}}^0 = \Omega_{S/\mathbf{Z}}^1$ is zero.

(b) Any perfect ring lies in QRSPerf . Two other interesting examples are $S = \mathcal{O}_{\mathbf{C}_p}/p$ and $S = \mathbf{F}_p[[T^{1/p^\infty}]]/(T-1)$.

(c) The category $\text{QRSPerf}^{\text{op}}$ with the quasi-syntomic topology forms a site (once again, only the existence of pull-backs is non obvious).

The following key result shows that quasi-regular semi-perfect rings form a basis of the quasi-syntomic topology on QSyn^{op} .

Proposition 1.8. *An \mathbf{F}_p -algebra A lies in QSyn if and only if there exists a quasi-syntomic cover $A \rightarrow S$, with $S \in \text{QRSPerf}$. Moreover, if $A \rightarrow S$ is a quasi-syntomic cover with $S \in \text{QRSPerf}$, all terms*

$$S^i := S \otimes_A S \otimes_A \cdots \otimes_A S \quad (i - \text{times})$$

of the Čech nerve S^\bullet lie in QRSPerf .

Proof. If there exists a quasi-syntomic cover $A \rightarrow S$, with $S \in \text{QRSPerf}$, the transitivity triangle :

$$L_{A/\mathbf{Z}} \otimes_A^{\mathbb{L}} S \rightarrow L_{S/\mathbf{Z}} \rightarrow L_{S/A}$$

shows that $L_{A/\mathbf{Z}} \otimes_A^{\mathbb{L}} S$ has Tor-amplitude in $[-1, 1]$, as the other two terms have Tor-amplitude in $[-1, 0]$ (because $S \in \text{QSyn}$ and because $A \rightarrow S$ is quasi-syntomic). By connectivity of the cotangent complex, this improves to $[-1, 0]$. As $A \rightarrow S$ is faithfully flat, we get that $A \in \text{QSyn}$.

Conversely, choose a surjective ring morphism :

$$F = \mathbf{F}_p[\{x_i\}_{i \in I}] \rightarrow A,$$

for some big enough index set I . Adjoining to F all p -power roots of the x_i , $i \in I$, one gets a perfect \mathbf{F}_p -algebra F_∞ . Base changing $F \rightarrow F_\infty$ along $F \rightarrow A$ gives a map

$$A \rightarrow S := F_\infty \otimes_F A.$$

The map $A \rightarrow S$ is a quasi-syntomic cover, being the base change of the quasi-syntomic cover $F \rightarrow F_\infty$. This easily implies (using the transitivity triangle for $\mathbf{Z} \rightarrow A \rightarrow S$) that $S \in \text{QSyn}$. Moreover F_∞ is perfect and surjects onto S .

The last assertion is left to the reader. □

The proposition implies that the restriction along the natural map

$$u : \text{QRSPerf}^{\text{op}} \rightarrow \text{QSyn}^{\text{op}}$$

induces an equivalence between sheaves on $\text{QRSPerf}^{\text{op}}$ and sheaves on QSyn^{op} with values in any reasonable¹ category \mathcal{C} . If \mathcal{F} is a \mathcal{C} -valued sheaf on $\text{QRSPerf}^{\text{op}}$, we call the associated sheaf on QSyn^{op} the *unfolding* of \mathcal{F} . It is explicitly computed as follows : if $A \in \text{QSyn}$, choose a quasi-syntomic cover $A \rightarrow S$, with $S \in \text{QRSPerf}$, and compute the totalization of the cosimplicial object $\mathcal{F}(S^\bullet)$ in \mathcal{C} .

¹In technical terms : any presentable ∞ -category.

Remark 1.9. In what follows, we will work with the category QSyn_R , for some fixed perfect ring R , formed by maps $R \rightarrow A$, with $A \in \text{QSyn}$, and similarly with QRSPerf_R . One can check that if $A \in \text{QSyn}_R$, $L_{A/R}$ has Tor-amplitude in $[-1, 0]$.

2. CONSTRUCTION OF THE FILTRATIONS ON THH AND ITS VARIANTS

Let R be a characteristic p perfect ring, fixed from now on. Let $A \in \text{QSyn}_R$. The goal is to endow $\text{THH}(A)$, $\text{TC}^-(A)$, $\text{TP}(A)$ and $\text{TC}(A)$ with complete exhaustive decreasing \mathbf{Z} -indexed multiplicative filtrations $\text{Fil}^* \text{THH}(A)$, $\text{Fil}^* \text{TC}^-(A)$, $\text{Fil}^* \text{TP}(A)$ and $\text{Fil}^* \text{TC}(A)$ (the *BMS filtrations*) such that

$$\hat{\Delta}_A := \text{gr}^0 \text{TC}^-(A) = \text{gr}^0 \text{TP}(A)$$

comes equipped with a complete decreasing \mathbf{N} -indexed multiplicative filtration $\mathcal{N}^{\geq *}\hat{\Delta}_A$ (the *Nygaard filtration*), with graded pieces $\mathcal{N}^*\hat{\Delta}_A$, together with natural isomorphisms :

$$\text{gr}^n \text{THH}(A) = \mathcal{N}^n \hat{\Delta}_A[2n] ; \text{gr}^n \text{TC}^-(A) = \mathcal{N}^{\geq n} \hat{\Delta}_A[2n] ; \text{gr}^n \text{TP}(A) = \hat{\Delta}_A[2n]$$

and :

$$\mathbf{Z}_p(n)(A) := \text{gr}^n \text{TC}(A) = \text{hofib}(\varphi - \text{can} : \mathcal{N}^{\geq n} \hat{\Delta}_A \rightarrow \hat{\Delta}_A).$$

Remarks 2.1. (a) The graded pieces $\mathbf{Z}_p(n)(A)$ are a priori spectra, but since $\mathbf{Z}_p(0)$ is the constant sheaf given by the Eilenberg-McLane spectrum of \mathbf{Z}_p and since all these graded pieces are module spectra over $\mathbf{Z}_p(0)(A)$, these spectra can be represented, non-canonically, by chain complexes.

(b) This filtration gives rise to the spectral sequence

$$E_2^{i,j} = \pi_{j-i}(\mathbf{Z}_p(-j)(A)) \implies \pi_{-i-j} \text{TC}(A)$$

alluded to in the introduction. The sheaves $\mathbf{Z}_p(n)$ are related the more classical logarithmic de Rham-Witt sheaves, as will be explained in the next talks.

The strategy to construct these filtrations is quite simple : one defines them *explicitely* at the level of quasi-regular semi-perfect rings ; one then uses *quasi-syntomic descent* to treat the case of general quasi-syntomic rings. In both cases, this remarkably reduces by some dévissages to understanding properties of the cotangent complex.

2.1. Computations for quasi-regular semi-perfect rings. We start by analyzing things for R itself.

Proposition 2.2. *Let R be a perfect \mathbf{F}_p -algebra. Then $\pi_* \text{THH}(R) \simeq R[u]$ is a polynomial algebra, with $u \in \pi_2 \text{THH}(R)$.*

Proof. We first prove that for any $R \rightarrow R'$, with R, R' perfect, the natural map

$$\text{THH}(R) \otimes_R^{\mathbb{L}} R' \rightarrow \text{THH}(R')$$

is an isomorphism. It is enough to check this after tensoring by \mathbf{Z} over $\text{THH}(\mathbf{Z})$ (because one can then argue by induction for $\otimes_{\text{THH}(\mathbf{Z})} \tau_{\leq n} \text{THH}(\mathbf{Z})$). Thus one needs to prove that :

$$\text{HH}(R) \otimes_R^{\mathbb{L}} R' \simeq \text{HH}(R').$$

Using the Hochschild-Kostant-Rosenberg filtration (HKR filtration), this amounts to prove that :

$$\wedge^i_R L_{R/\mathbf{Z}} \otimes^{\mathbb{L}}_R R' \simeq \wedge^i_{R'} L_{R'/\mathbf{Z}},$$

which is easily deduced from Example 1.4. This base change property reduces us to prove the proposition for $R = \mathbf{F}_p$; in this case, this is the content of Bökstedt's theorem. \square

Proposition 2.3. *One can find generators $u \in \pi_2 \mathrm{TC}^-(R)$, $v \in \pi_{-2} \mathrm{TC}^-(R)$ and $\sigma \in \pi_2 \mathrm{TP}(R)$ such that :*

$$\pi_* \mathrm{TC}^-(R) \simeq W(R)[u, v]/(uv - p) ; \pi_* \mathrm{TP}(R) \simeq W(R)[\sigma, \sigma^{-1}]$$

and such that the map induced on homotopy groups by

$$\varphi^{h\mathbb{T}} : \mathrm{TC}^-(R) = \mathrm{THH}(R)^{h\mathbb{T}} \rightarrow \mathrm{TP}(R) = (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}}$$

is the $\varphi_{W(R)}$ -linear map sending u to σ and v to $p\sigma^{-1}$, and such that the map induced on homotopy groups by

$$\mathrm{can} : \mathrm{TC}^-(R) \rightarrow \mathrm{TP}(R)$$

is the linear map sending u to $p\sigma$ and v to σ^{-1} .

Proof. We simply describe $\pi_0 \mathrm{TC}^-(R)$, which is the hardest part, and refer the reader to [2, §6] for the rest. Because $\pi_* \mathrm{THH}(R)$ is concentrated in even degrees and has trivial \mathbb{T} -action, the homotopy fixed point spectral sequence :

$$E_2^{i,j} = H^i(\mathbb{T}, \pi_{-j} \mathrm{THH}(R)) \implies \pi_{-i-j} \mathrm{TC}^-(R)$$

degenerates. In particular, one can lift $u \in \pi_2 \mathrm{THH}(R)$ to an element (still denoted) $u \in \pi_2 \mathrm{TC}^-(R)$ and the natural generator of $H^2(\mathbb{T}, \pi_0 \mathrm{THH}(R))$ to $v \in \pi_{-2} \mathrm{TC}^-(R)$. The degeneracy of the spectral sequences also provides a descending complete \mathbb{N} -indexed multiplicative filtration on $\pi_0 \mathrm{TC}^-(R)$ such that

$$\mathrm{gr}^i \mathrm{TC}^-(R) = \pi_{2i} \mathrm{THH}(R) \simeq R,$$

for $i \geq 0$. In particular, the map

$$\pi_0 \mathrm{TC}^-(R) \rightarrow \pi_0 \mathrm{THH}(R) = R$$

makes $\pi_0 \mathrm{TC}^-(R)$ a pro-infinitesimal thickening of R . By the universal property of $W(R)$, this gives a unique map $W(R) \rightarrow \pi_0 \mathrm{TC}^-(R)$, with

$$\mathrm{im}(pW(R)) \subset \mathrm{Fil}^1 \pi_0 \mathrm{TC}^-(R) = \ker(\pi_0 \mathrm{TC}^-(R) \rightarrow R).$$

By multiplicativity of the filtration, one has

$$\mathrm{im}(p^i W(R)) \subset \mathrm{Fil}^i \pi_0 \mathrm{TC}^-(R),$$

for all $i \geq 1$. Proving that the map $W(R) \rightarrow \pi_0 \mathrm{TC}^-(R)$ is an isomorphism can thus be checked on graded pieces, i.e. by showing that certain maps from R to R are isomorphisms, which readily reduces by base change to the case $R = \mathbf{F}_p$. In this case, we know by [1, Lem. IV.4.7] that the images of p and uv in $H^2(\mathbb{T}, \pi_2 \mathrm{THH}(\mathbf{F}_p))$ are the same. By multiplicativity, the images of p^i and $u^i v^i$ in

$H^{2i}(\mathbb{T}, \pi_{2i} \mathrm{THH}(\mathbf{F}_p))$ are the same. Hence all the graded maps are isomorphisms. Up to modifying u by a unit, we can also arrange that $uv = p$ in $\pi_0 \mathrm{TC}^-(R)$. \square

Now we can turn to quasi-regular semi-perfect rings.

Theorem 2.4. *Let $S \in \mathrm{QRSPerf}_R$. Then :*

- (1) $\pi_* \mathrm{THH}(S)$ only lives in even degrees.
- (2) Let $i \in \mathbf{Z}$. Multiplication by $u \in \pi_2 \mathrm{THH}(R)$ gives an injective map :

$$\pi_{2i-2} \mathrm{THH}(S) \rightarrow \pi_{2i} \mathrm{THH}(S)$$

and this endows $\pi_{2i} \mathrm{THH}(S)$ with a functorial finite increasing filtration with graded pieces $\wedge_S^j L_{S/R}[-j]$, for $0 \leq j \leq i$ in increasing order.

Proof. We start by noting for any R -algebra A , we have a \mathbb{T} -equivariant fiber sequence

$$\mathrm{THH}(A)[2] \rightarrow \mathrm{THH}(A) \rightarrow \mathrm{HH}(A/R) \quad (*)$$

(see [2, Th. 6.7]). We will first apply this when A is a quasi-smooth R -algebra. Then, by the universal property of the de Rham complex, the natural antisymmetrisation map

$$\Omega_{A/R}^1 = \Omega_{A/\mathbf{Z}}^1 \rightarrow \pi_1 \mathrm{HH}(A) = \pi_1 \mathrm{THH}(A)$$

(the first equality comes from the fact that R is perfect) extends to a map of graded A -algebras $\Omega_{A/R}^* \rightarrow \pi_* \mathrm{THH}(A)$. Using the HKR filtration, one sees that the composite of this map with the map $\pi_* \mathrm{THH}(A) \rightarrow \pi_* \mathrm{HH}(A/R)$ is an isomorphism. Thus the long exact sequence on homotopy groups induced by the fiber sequence (*) splits in short exact sequences, for all i :

$$0 \rightarrow \pi_{i-2} \mathrm{THH}(A) \rightarrow \pi_i \mathrm{THH}(A) \rightarrow \pi_i \mathrm{HH}(A/R) \simeq \Omega_{A/R}^i \rightarrow 0.$$

Therefore, the natural map

$$\Omega_{A/R}^* \otimes_R \pi_* \mathrm{THH}(R) \rightarrow \pi_* \mathrm{THH}(A)$$

has to be an isomorphism. This proves that on the category of quasi-smooth algebras over R , the Postnikov filtration on THH is a complete decreasing \mathbf{N} -indexed multiplicative filtration Fil_P^* with graded pieces

$$\mathrm{gr}_P^n \mathrm{THH}(-) \simeq \bigoplus_{0 \leq i \leq n, i-n \text{ even}} \Omega_{-/R}^i[n].$$

By left Kan extension, we get a complete decreasing \mathbf{N} -indexed multiplicative filtration Fil_P^* on THH over the category of all R -algebras, with graded pieces

$$\mathrm{gr}_P^n \mathrm{THH}(-) \simeq \bigoplus_{0 \leq i \leq n, i-n \text{ even}} \wedge^i L_{-/R}[n].$$

Now we apply this to our quasi-regular semi-perfect ring S over R . By Remark 1.7 (a) and induction on i , $\wedge^i L_{S/R}$ has Tor-amplitude in $[-i, -i]$ and thus lives in homological degree i . Hence, for any n , $\mathrm{gr}_P^n \mathrm{THH}(S)$ only lives in even degrees. This implies (1), by completeness of the filtration.

To prove (2), we use the fiber sequence (*) for $A = S$. As the homotopy groups of all terms are in even degrees (we just proved it for $\mathrm{THH}(S)$ and it is easily verified for $\mathrm{HH}(S/R)$ using the HKR filtration), the long exact sequence on homotopy groups splits in short exact sequences :

$$0 \rightarrow \pi_{2i-2}\mathrm{THH}(S) \rightarrow \pi_{2i}\mathrm{THH}(S) \rightarrow \pi_{2i}\mathrm{HH}(S/R) \rightarrow 0,$$

for all i . This provides the desired filtration on $\pi_{2i}\mathrm{THH}(S)$, as (by the HKR filtration), $\pi_{2i}\mathrm{HH}(S/R) = \wedge_S^i L_{S/R}[-i]$. \square

Remark 2.5. The filtration Fil_P^* was only introduced as an auxiliary tool ; as we will see, the interesting filtration is the one defined by (2) of the proposition, which comes from the (double speed) Postnikov filtration. That they differ is explained by the fact that the Postnikov filtration on THH over $\mathrm{QRSPerf}$ is not (the restriction to $\mathrm{QRSPerf}$ of) the left Kan extension of the Postnikov filtration on THH over quasi-smooth R -algebras.

Theorem 2.6. *Let $S \in \mathrm{QRSPerf}_R$.*

(1) *The homotopy fixed point and Tate spectral sequences computing $\mathrm{TC}^-(S)$ and $\mathrm{TP}(S)$ degenerate. Both $\pi_*\mathrm{TC}^-(S)$ and $\pi_*\mathrm{TP}(S)$ live in even degrees.*

(2) *The degenerate homotopy fixed point and Tate spectral sequences endow*

$$\hat{\Delta}_S := \pi_0\mathrm{TC}^-(S) \underset{\mathrm{can}}{\simeq} \pi_0\mathrm{TP}(S)$$

with the same descending complete \mathbf{N} -indexed filtration $\mathcal{N}^{\geq}\hat{\Delta}_S$, with graded pieces denoted by $\mathcal{N}^*\hat{\Delta}_S$.*

(3) *One has, for any n , natural identifications :*

$$\pi_{2n}\mathrm{THH}(S) = \mathcal{N}^n\hat{\Delta}_S ; \pi_{2n}\mathrm{TC}^-(S) = \mathcal{N}^{\geq n}\hat{\Delta}_S ; \pi_{2n}\mathrm{TP}(S) = \hat{\Delta}_S.$$

(4) *One has a natural isomorphism of R -algebras*

$$\hat{\Delta}_S/p \simeq \widehat{L}\Omega_{S/R}$$

(the right hand side is the Hodge-completed derived de Rham complex of S over R) and $\hat{\Delta}_S$ is p -torsion free.

Proof. As $\pi_*\mathrm{THH}(S)$ only lives in even degrees, the first three points are easy. The proof of (4) relies on the fiber sequence used in the proof of Theorem 2.4 and the identification of $\pi_0\mathrm{HC}^-(S)$ as the Hodge-completed derived de Rham complex $\widehat{L}\Omega_{S/R}$ (whose proof uses quite subtle arguments about filtered derived categories and is given in [2, Prop. 5.14]). \square

2.2. The filtrations. We start by reminding the reader that the cotangent complex (and its wedge powers), the Hodge-completed derived de Rham complex, THH , TC^- , TP and TC are all fpqc sheaves. This was proved in the last talk by reduction to the case of the cotangent complex and will be crucial for us.

We first explain the construction of the BMS filtration for THH on quasi-syntomic rings. As promised, this is done in two steps. By Theorem 2.4, if $S \in \mathrm{QRSPerf}_R$ and $i \in \mathbf{Z}$, $\pi_{2i}\mathrm{THH}(S)$ has a functorial finite increasing filtration

with graded pieces $\wedge_S^j L_{S/R}[-j]$, for $0 \leq j \leq i$ in increasing order. In other words, if $S \in \text{QRSPerf}_R$, the double speed Postnikov filtration endows the spectrum $\text{THH}(S)$ with a functorial complete descending \mathbf{Z} -indexed \mathbb{T} -equivariant filtration $\text{Fil}^* \text{THH}(S)$ such that $\text{gr}^i \text{THH}(S)$ is canonically an S -module spectrum (with trivial \mathbb{T} -action) admitting a finite increasing filtration with graded pieces given by $\wedge_S^j L_{S/R}[2i-j]$, $0 \leq j \leq i$. This is our BMS filtration on QRSPerf_R , and the end of the first step.

The second step is quasi-syntomic descent. We recalled that THH on $\text{QSyn}_R^{\text{op}}$ is the unfolding of its restriction to $\text{QRSPerf}_R^{\text{op}}$. The last paragraph demonstrates that the double speed Postnikov filtration on THH over $\text{QRSPerf}_R^{\text{op}}$ and the filtration on its graded pieces unfold to $\text{QSyn}_R^{\text{op}}$: indeed, wedge powers of the cotangent complex satisfy descent. Therefore, we see that for any $A \in \text{QSyn}_R$, the spectrum $\text{THH}(A)$ admits a functorial complete descending \mathbf{Z} -indexed \mathbb{T} -equivariant filtration $\text{Fil}^* \text{THH}(A)$ such that $\text{gr}^i \text{THH}(A)$ is canonically an A -module spectrum (with trivial \mathbb{T} -action) admitting a finite increasing filtration with graded pieces given by $\wedge_A^j L_{A/R}[2i-j]$, $0 \leq j \leq i$.

The same game can be played with Theorem 2.6, to construct the Nygaard filtration: the sheaf $\hat{\Delta}_-$ and its filtration $\mathcal{N}^{\geq *}\hat{\Delta}_-$ on $\text{QRSPerf}_R^{\text{op}}$ unfold² to $\text{QSyn}_R^{\text{op}}$, since by Theorem 2.6, one has, for all $S \in \text{QRSPerf}_R$ and all n ,

$$\mathcal{N}^n \hat{\Delta}_S \simeq \pi_{2n} \text{THH}(S)[-2n],$$

and we just checked that the right hand side unfolds to a sheaf on QSyn^{op} .

This unfolding defines $(\hat{\Delta}_-, \mathcal{N}^{\geq *}\hat{\Delta}_-)$ on $\text{QSyn}_R^{\text{op}}$, and one has, for all $A \in \text{QSyn}_R$ and all n ,

$$\mathcal{N}^n \hat{\Delta}_A \simeq \pi_{2n} \text{THH}(A)[-2n],$$

as well as a natural identification of E_∞ - R -algebras $\hat{\Delta}_A/p \simeq \widehat{L\Omega}_{A/R}$.

The same kind of arguments apply to construct the sought after filtrations on TC^- , TP and TC on QSyn_R : cf. [2, Prop. 7.13].

REFERENCES

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- [2] B. Bhatt, M. Morrow and P. Scholze, *Topological Hochschild homology and integral p -adic Hodge theory*, Preprint (2018).

²The most economic way to do this is to see them as defining a sheaf with values in the complete filtered derived category of $W(R)$ -modules.