Fourier-Whittaker expansions: from analysis on global fields to geometry on local fields

Talk in Lille

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1 Fourier-Whittaker expansions: global theory

An important tool for the study of automorphic forms and their L-functions is their Fourier expansion / Whittaker coefficients.

Let's recall this briefly for GL_2 (would work similarly for GL_n). Let E = global field, let f be an automorphic form for GL_2/E.

For any $g \in GL_2(\mathbb{A}_E)$, can write the Fourier expansion of : $N(E)\backslash N(\mathbb{A}_E) \simeq E\backslash \mathbb{A}_E \to \mathbb{C}$, $n \mapsto f(ng)$.

Get, when f is cuspidal, that

$$\forall g \in GL_2(\mathbb{A}_E),\quad f(g) = \sum_{\gamma \in E^\times} W_{f,\psi}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right) \qquad (*)$$

with $\psi : E\backslash \mathbb{A}_E \to \mathbb{C}^\times$ non-trivial character (used to define the FT)

and

$$W_{f,\psi}(g) = \int_{N(E)\backslash N(\mathbb{A}_E)} f(ng)\,\psi^{-1}(n)\,dn \quad \text{"first Fourier coefficient of } f\text{"}$$

$$\in \operatorname{Fun}(GL_2(\mathbb{A}_E), \mathbb{C})^{(N(\mathbb{A}_E),\psi)} \quad \text{"space of Whittaker functions"}$$

The L-function of f can then be described as the Mellin transform of $W_{f,\psi}$. Useful perspective to generalize construction of L-functions to automorphic reps of GL_2, GL_n (Godement, Jacquet-Langlands).

Note that for any W Whittaker function, the formula $(*)$ with $W_{f,\psi}$ replaced by W gives a cuspidal function $f_W : M(E)\backslash GL_2(\mathbb{A}_E) \to \mathbb{C}$.

with $M = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \subseteq GL_2$ mirabolic subgroup.

$\to$ Can also be useful to construct automorphic forms.

<u>Example</u> (Shintani) Let $\gamma = (\gamma_v)$ collection of conjugacy classes in $GL_n(\mathbb{C})$ indexed by the places of E. To this one can attach a Whittaker function W_γ s.t. $f_{W_\gamma} \in \mathrm{Fun}_{cusp}(M(E)\backslash GL_2(\mathbb{A}_E), \mathbb{C})$ is right-invariant by maximal compact and Hecke eigenform with Hecke eigenvalue at v determined explicitly by γ_v. Moreover, unique with these properties.

Using this, the unramified part of the "Galois → automorphic" direction of the Langlands conjecture can be rephrased as:

If there exists $\sigma : \mathrm{Gal}_E \to GL_n(\overline{\mathbb{Q}}_\ell)$ $\ell \neq p$ irreducible, everywhere unramified s.t. $\forall$ place v, $\gamma_v = \sigma(\mathrm{Frob}_v)$, then the function $f_\sigma := f_{W_\gamma}$ is left-invariant by $GL_2(E)$ (hence a cuspidal Hecke eigenform for σ).

One can prove this conjecture by geometric means, when $E =$ function field (Drinfeld for GL_2)

X smooth projective geom. curve over $k = \mathbb{F}_q$ s.t. $E = k(X)$. Let:

- For $i \geq 0$, $\mathcal{C}_i$ = moduli stack of flat coherent sheaves of generic rank i.
- For $\mathcal{E}$ = universal coherent sheaf on $\mathcal{C}_1 \times X$,

let $\mathcal{V} = \underline{\mathrm{Hom}}(\mathcal{O}_X, \mathcal{E}) \supseteq \mathcal{V}^\circ$ = substack where the section is injective.

vector bundle on (an open substack of) $\mathcal{C}_1$

- let $\mathcal{V}^\vee = \underline{\mathrm{Ext}}^1(\mathcal{E}, \omega_X)$

$$\cup\!\!\!\shortmid$$

$\mathrm{Bun}_2' =$ locus where the extension is a rank 2 v.b. (i.e., no torsion) = moduli of pairs $(\mathcal{F}, s)$, $\mathcal{F}$ rank 2 vb, $s : \omega_X \hookrightarrow \mathcal{F}$.

Note that, as notation suggests, $\mathcal{V}$ and $\mathcal{V}^\vee$ are dual vector bundles over $\mathcal{E}_1$. The duality pairing is given by

$$\mathcal{V} \times \mathcal{V}^\vee \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{O}, \omega_X) \simeq \mathbb{A}^1$$

The character ψ gives Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1$ and a Fourier equivalence: (Deligne, Laumon)

$$D_{\acute{e}t}(\mathcal{V}, \overline{\mathbb{Q}}_\ell) \simeq D_{\acute{e}t}(\mathcal{V}^\vee, \overline{\mathbb{Q}}_\ell).$$

(Here, $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$, induces $\psi : E\backslash \mathbb{A}_E \to \overline{\mathbb{Q}}_\ell^\times$ by $(a_v) \mapsto \psi\left(\sum_v \mathrm{tr}_{k_v/k}(\mathrm{Res}(a_v\omega))\right)$ $\omega \in \Omega^1_{E/k} \setminus \{0\}$)

Now, σ can be seen as a lisse sheaf on X (since $\pi_1(X) = \mathrm{Gal}_E^{unr}$).

Even better, can be promoted to $\mathcal{L}_\sigma \in D_{\acute{e}t}(\mathcal{E}_0, \overline{\mathbb{Q}}_\ell)$.

Have a map $\mathcal{V}^\circ \longrightarrow \mathcal{E}_0$ (taking cokernel of the section).

Can pullback $\mathcal{L}_\sigma$ along it and $!$-extend to $\mathcal{V}$.

Then Fourier transform and restrict to Bun_2'.

Get $\mathrm{Aut}'_\sigma \in D(\mathrm{Bun}'_2, \overline{\mathbb{Q}}_\ell)$.

By taking trace of Frobenius, geometrizes the function f_σ mentioned above. Proving the conjecture amounts to showing that Aut'_σ descends to a "Hecke eigensheaf" on Bun_2 along the forgetful map $\mathrm{Bun}'_2 \to \mathrm{Bun}_2$, $(\mathcal{F}, s) \mapsto \mathcal{F}$.

2 A local geometric analogue: ℓ-adic coefficients

In recent years, spectacular advances in using such geometric methods in the local setting.

(Fargues-Scholze, Zhu)

From now on, E = local non-arch. field.

Attached to E, there is for each perfectoid space S over its residue field, a "family of curves"

$X_{S,E}$ ("relative Fargues-Fontaine curve")

Can thus define:

- $\mathrm{Div}^1 : S \mapsto \left\{ \begin{array}{c} \text{effective relative Cartier} \\ \text{divisors of degree } 1 \text{ on } X_{S,E} \end{array} \right\}$

ℓ-adic reps of $W_E \hookrightarrow$ local ℓ-adic sheaves on Div^1

- $\mathrm{Bun}_2 : S \mapsto$ groupoid of rank 2 vector bundles on $X_{S,E}$.

$|\mathrm{Bun}_2| =$

$\vdots$ $\quad \bullet\ \mathcal{O}(-2)\oplus\mathcal{O}(2)$

$\bullet\ \mathcal{O}(-1)\oplus\mathcal{O}(1)$ $\quad \bullet\ \mathcal{O}\oplus\mathcal{O}(1)$ $\quad$ specialization relations

semi-stable locus, open $\rightarrow \ \cdots \quad \bullet\ \mathcal{O}(-\tfrac{1}{2}) \circlearrowleft D_E^\times \quad \bullet\ \mathcal{O}^{\oplus 2} \circlearrowleft GL_2(E) \quad \bullet\ \mathcal{O}(\tfrac{1}{2}) \circlearrowleft D_E^\times \quad \bullet\ \mathcal{O}(1)^{\oplus 2} \circlearrowleft GL_2(E) \ \cdots$

$\rightsquigarrow$ $D\left(\begin{array}{c}\text{smooth } \overline{\mathbb{Q}}_\ell\text{-reps of}\\ GL_2(E)\end{array}\right)$, $D\left(\begin{array}{c}\text{smooth } \overline{\mathbb{Q}}_\ell\text{-reps of}\\ D_E^\times\end{array}\right)$ $\underset{\text{fully faithful}}{\subseteq} D_{\text{ét}}(\mathrm{Bun}_2, \overline{\mathbb{Q}}_\ell)$.

<u>Question</u>: What does the above construction of Drinfeld(-Laumon) give in this context?

Key point: understand what the correct Fourier transform is in this setting. Earlier, had $\underline{\mathrm{Ext}}^1(\mathcal{O}_X, \omega_X) \underset{\text{Serre duality}}{\simeq} \mathbb{A}^1$ with its Artin-Schreier sheaf attached to ψ.

$\rightsquigarrow$ Fourier transform for sheaves of sections of flat coherent sheaves on X.

Jt. work with Anschütz: When X is the Fargues-Fontaine curve, replace ω_X by $\mathcal{O}_X$. The <u>stack</u> of extensions of $\mathcal{O}_X$ by $\mathcal{O}_X$ is $[* / \underline{E}]$. Smooth character $\psi: E \to \overline{\mathbb{Q}}_\ell^*$ tautologically gives rise to rank 1 line sheaf on $[* / \underline{E}]$.

$\rightsquigarrow$ Fourier transform for Banach-Colmez spaces (= sheaves of sections of "flat" coherent sheaves on the FF curve).

Using this, can rerun the above construction in this new geometric setting.

We therefore fix $\sigma: W_E \to GL_2(\overline{\mathbb{Q}}_\ell)$ continuous irreducible Weil representation. Hope to produce $\mathrm{Aut}'_\sigma \in D(\mathrm{Bun}'_2, \overline{\mathbb{Q}}_\ell)$ descending to $\mathrm{Aut}_\sigma \in D(\mathrm{Bun}_2, \overline{\mathbb{Q}}_\ell)$ whose restriction to the semi-stable locus relates to σ via LLC + JLC.

To see that this is not unreasonable, can specialize to the fiber of $\mathrm{Bun}'_2 \to \mathrm{Bun}_2$ over the locus where the v.b. $\mathcal{F}$ is isomorphic to $\mathcal{O}^{\oplus 2}$ or $\mathcal{O}(\frac{1}{2})$:

- Case $\mathcal{F} \simeq \mathcal{O}^{\oplus 2}$. The fiber is $[*/M(E)]$ and the map the natural map $[*/M(E)] \to [*/Gl_2(E)]$

Diagram:

$$\begin{tikzcd}
{[*/M(E)]} \arrow[d] \arrow[r, phantom, "\simeq"] & {[*/M(E)]} \arrow[dr] & & {[E/E^\times]} \arrow[dl] \arrow[r, phantom, "\supseteq"] & E^\times/E^\times \simeq * . \\
{[*/Gl_2(E)]} & & {[*/E^\times]} & &
\end{tikzcd}$$

(1)

$\to$ the sheaf on $[*/M(E)]$ produced by the Drinfeld construction is $c\text{-}ind_1^{M(E)} \psi$

$=$ Kirillov model of any supercuspidal rep!

- Case $\mathcal{F} \simeq \mathcal{O}(\frac{1}{2})$.

Diagram:

$$\begin{tikzcd}
{\left[\frac{BC(\mathcal{O}(\frac{1}{2}))\setminus 0}{D_E^\times}\right]} \arrow[d] \arrow[r, phantom, "\subseteq"] & {\left[\frac{BC(\mathcal{O}(-1))}{E^\times}\right]} \arrow[r] & {[*/E^\times]} & {\left[\frac{BC(\mathcal{O}(1))}{E^\times}\right]} \arrow[l] \arrow[r, phantom, "\supseteq"] & Div^1 \\
{[*/D_E^\times]} & & & &
\end{tikzcd}$$

(2)

Hence, geometric construction from σ of a $D_E^\times$-equivariant sheaf on $BC(\mathcal{O}(\frac{1}{2}))\setminus 0$ which should be constant as a sheaf, given by the smooth rep of $D_E^\times$ attached to σ by LLC + JLC.

→ In particular, should have finite rank and can at least check it has the correct rank (Deligne, Carayol) using the adic GOS formula (Ramero, Huber).

Rk: Construction of $\mathcal{L}_\sigma$ still has to be done, and needs genuinely new ideas to prove descent to Bun_2.

③ A local geometric analogue: towards mod p coefficients? (in progress)

Assume $E \supseteq \mathbb{Q}_p$.

The recent PhD thesis of Mann allows to define a category $D(\mathrm{Bun}_2)$ of "mod p sheaves on Bun_2". This category has a behaviour which is in between étale and coherent sheaves.

mod p (φ, Γ)-modules (Lubin-Tate version) $\hookrightarrow$ dualizable objects (in deg 0) in $D(\mathrm{Div}^1)$.

Is there an analogue of Drinfeld construction in this setting too? (Question independently asked and studied by Fargues)

Fourier theory has to work differently. Geometric objects stay the same (Banach-Colmez spaces), but coefficients have changed, and it now seems the kernel of the FT should be the rank 1 object in $D([*/BC(\mathcal{O}(d))])$, $d=[E:\mathbb{Q}_p]$, induced by $BC(\mathcal{O}(d)) \simeq \widetilde{\mathbb{G}}_m$.

Hence, in Drinfeld construction, the diagram appearing when specializing to the fiber of Bun_2' over the locus where the bundle is isomorphic to $\mathcal{O}(d)^{\oplus 2}$ now looks like a mix between (1) and (2):

$$\begin{tikzcd}
{[*/M(E)]} \arrow[d] & {[*/M(E)]} \arrow[l, phantom, "\simeq"] \arrow[dr] & & {\left[\frac{BC(\mathcal{O}(d))}{E^\times}\right] \supseteq \mathrm{Div}^d} \arrow[dl] \\
{[*/GL_2(E)]} & & {[*/E^\times]} &
\end{tikzcd}$$

TO $\sigma : \mathrm{Gal}_E \to GL_2(\overline{\mathbb{F}}_p)$ irreducible

$\mapsto$ irreducible rank 2 (φ,Γ)-module seen in Div^1, can thus attach a rep of the mirabolic subgroup. This is (should be..) Colmez construction when $d=1$, i.e. $E=\mathbb{Q}_p$.

④ Whittaker models

Global / local Whittaker models: embedding of a generic (e.g. tempered) representation in the space of Whittaker functions (seen as a rep of the group by right-translation action). Dually: any such representation admits a surjection from the compact induction of ψ from upper unipotent to GL_n =: the Whittaker rep.

Geometric Langlands perspective on this: for any choice of base (global/local) and coefficients (ℓ-adic, mod p, ...), should have an equivalence

$$D(\mathrm{Bun}_n) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}\left(\begin{array}{c}\text{moduli of}\\ L\text{-parameters}\\ \text{for } GL_n\end{array}\right)$$

compatible with Hecke actions and many other structures. We do not try to make the statement more precise here.

Let us point out that if we restrict to the "tempered part" on the LHS (in particular, disregard anything non generic), can just pretend that one considers QCoh on the RHS.

Let $\mathcal{W}$ = <u>Whittaker sheaf</u> = image of structure sheaf on the stack of L-parameters.

Since the structure sheaf maps surjectively on "the" skyscraper sheaf at any point of the stack of L-parameters, $\mathcal{W}$ functions as a geometric incarnation of the Whittaker representation.

In fact, $\exists$ general guess for what $\mathcal{W}$ should be: there for GL_2 for simplicity. Consider:

$$\mathcal{N} = \begin{matrix}\text{stack of extensions} \\ \text{of } \mathcal{O}_X \text{ by } \omega_X\end{matrix} \xrightarrow[f]{} Bun_2$$

(send the extension to the underlying bundle)

Note that in any of the three cases discussed above, $\mathcal{N}$ is where the kernel of the Fourier transform lives. Its image by $f_!$ should be $\mathcal{W}$.

To finish, let us see what this says in the local setting:

- local setting, ℓ-adic coefficients: gives $\mathcal{W} = i_{1,!}\left(\text{c-ind}_{N(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \psi\right)$. In particular, fiber at $\mathrm{Bun}_2^1 = [*/GL_2(\mathbb{Q}_p)] \underset{\text{open}}{\subseteq} \mathrm{Bun}_2$ is the Whittaker representation.

- local setting, mod p coefficients: in this context, there is as far as as I know no theory of the Whittaker model (basic problem: no ψ!). This is "explained" by the shape of $\mathcal{W}$: it is supported on the stratum given by $\mathcal{O} \oplus \mathcal{O}(1)$, hence is zero in restriction to Bun_2^1.