LECTURE III : THE CASE OF CRYSTALLINE REPRESENTATIONS

$\boxed{G = GL_2(\mathbb{Q}_p)}$

In this lecture, I want to explain what happens for 2-dim'l crystalline reps of $\mathcal{G}_{\mathbb{Q}_p}$. Recall that via the D_{cris} functor of Fontaine, these are the same as 2-dim'l admissible filtered φ-modules over $\mathbb{Q}_p$. These can be classified:

Fix $k \geq 2$ integer, $a, b \in \mathfrak{m}_L$ st. $v_p(ab) = k-1$, $\underline{v_p(a) \geq v_p(b)}$

Then: $D_{cris}(a,b) = Le_a \oplus Le_b$

$$\text{if } a \neq b \left\{ \begin{array}{l} \text{with } \varphi(e_a) = a^{-1} e_a, \quad \varphi(e_b) = b^{-1} e_b. \\ \& \ \mathrm{Fil}^i(D(a,b)) = \begin{cases} D(a,b) & \text{if } i \leq -(k-1) \\ L(e_a + e_b) & \text{if } -(k-1) < i \leq 0 \\ 0 & \text{if } i > 0 \end{cases} \end{array} \right.$$

$$\text{if } a = b \left\{ \begin{array}{l} \varphi(e_a) = a^{-1} e_a, \quad \varphi(e_b) = b^{-1}(e_b - e_a) \\ \mathrm{Fil}^i = \begin{cases} D(a,b) & \text{—} \\ Le_b & \text{—} \\ 0 & \text{—} \end{cases} \end{array} \right.$$

is a filtered admissible φ-module $/\mathbb{Q}_p$, and they can all be obtain in this way. (the abs. irred ones)

From now on, assume $\underbrace{a \neq b}$, $\underbrace{a \neq p^{\pm 1} b}$

technical hyp
(important for the specific methods to be used)

In particular, we see that given $D_{a,b}$ (without its Hodge filtration) as an isocrystal there exist, for each $k \geq 2$, exactly one admissible filtat° of wts $0, k-1$.

Hence if we believe that Breuil's philosophy is correct, there must exist one interesting completion of $LL(D_{a,b}) \otimes \mathrm{Sym}^{k-2}$ $\forall k \geq 2$.

There is an obvious candidate for that, which will turn out to be the good one, as I now explain: $:= \Pi_{a,b}$.

Def III.1 W locally convex top vs with a C^0 G-action.
U unitary Banach rep of G. Say that a given linear C^0 G-eq map: $W \to U$ realizes U as a universal unitary completion of W if the map $\mathrm{Hom}_G^{C^0}(U, W') \to \mathrm{Hom}_G^{C^0}(W, W')$ is a bijection.
(U if it exists is unique up to unique iso)
Call it $\hat{W}$. This construction is simple, functorial, but <u>badly behaved</u>:
$\hat{W}$ does not always exist. For ex, assume W to be top irred. Then ^ it can be zero
the map $W \to \hat{W}$ is non zero (because it has dense image), hence injective: pulling back a G-inv norm on $\hat{W}$, you get a G-inv norm on W. <u>In particular, central character of W has to be unitary.</u>
Nice <u>example</u>: $\mathbb{Z}_p$ acting on $LA(\mathbb{Z}_p, L)$ by translations; UUC does not exist! cf [CD], Prop VII.3
Anyway, here is a criterion for $\hat{W}$ to exist:
<u>Prop</u> III.2: W admits a UUC iff the set of commensurability classes of G-stable open lattices in W has a minimal element.
(Proof is easy)
<u>Prop</u> III.3: W fin. gen G-rep, equipped with its finest local. conv top. The completion of W wrt to any finite type lattice of W (= G-inv lattice which is fin. gen. as an $\mathcal{O}_L[G]$-module) is a UUC for W.
(<u>Pf</u>: Any lattice in W is open (loc. convex top.) & contains ~~an~~ a f.g. sublattice (since V is fin gen). Apply Prop III.2)
Prop III.3 applies to our rep $\pi_{a,b}$ (which is irreducible).
$\leadsto$ gives $\widehat{\pi_{a,b}}$.
Goal of this lecture:
<u>Th</u> III.4 (Berger-Breuil) The G-rep $\widehat{\pi_{a,b}}$ is a non zero abs irred rep in Ban(G).
• Even the fact that $\widehat{\pi_{a,b}} \neq 0$ is very hard!!

LLC says that $\pi_{a,b} \cong \mathrm{Ind}_B^G(\alpha \otimes \beta|\cdot|^{-1})^{\text{smooth}} \otimes \mathrm{Sym}^{k-2}(L^2)$

where $\alpha, \beta : \mathbb{Q}_p^\times \to L^\times$

$$\alpha(x) = a^{-v_p(x)}$$
$$\beta(x) = b^{-v_p(x)}.$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(z) = (-cz+a)^{k-2} P\left(\frac{dz-b}{a-cz}\right)$

One should think of element of $\pi_{a,b}$ as functions $f : \mathbb{Q}_p \to L$ locally polynomial of deg $\leq k-2$ & st

$x \mapsto \sigma(x)\, x^{k-2} f(1/x)$ is loc. pol of deg $\leq k-2$, near 0, with $\sigma = \frac{\beta}{\alpha|\cdot|}$, and the action of G given by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(z) = \alpha(ad-bc)\,(\beta\alpha^{-1})\,(a-cz)^{k-2} \underbrace{|a-cz|\,(a-cz)}_{\sigma(z)} f\left(\frac{dz-b}{a-cz}\right)$$

(to see this, send $F \in \mathrm{Ind}_B^G(\alpha \otimes \beta|\cdot|^{-1})$ to the loc cst $f : z \mapsto F\begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$)

Let $B(a,b)$ be the following Banach space: underlying top vs = functions $f : \mathbb{Q}_p \to L$, st. $f_{|\mathbb{Z}_p} \in \mathcal{C}^{v(a)}(\mathbb{Z}_p, L)$ & $\sigma(z)\, z^{k-2} f(1/z)_{|\mathbb{Z}_p - \{0\}}$ extends to ~~$\mathbb{Z}_p$~~ an element of $\mathcal{C}^{v(a)}(\mathbb{Z}_p, L)$.

Action of G: $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(z) = \alpha(ad-bc)\,\sigma(a-cz)\,(a-cz)^{k-2} f\left(\frac{dz-b}{a-cz}\right)$

(continuous: not so easy...)

Let $L(a,b)$ be the closure of the subspace of $B(a)$ generated by $z \mapsto z^j$, $z \mapsto \sigma(z-\lambda)(z-\lambda)^{k-2-j}$, $\lambda \in \mathbb{Q}_p$, $0 \leq j < v(a)$.

This subspace is stable by G.

(Recall: $a_n(f) = \sum_0^n (-1)^i \binom{n}{i} f(n-i)$; f is in $C^r(\mathbb{Z}_p)$ if $n^r |a_n(f)| \xrightarrow[n\to\infty]{} 0$)

Finally, let $\Pi(a,b) = B(a,b)/L(a,b)$.

Prop III.5 ([BB], Th. 4.3.1) As G-repr, $\widehat{\pi_{a,b}} \simeq \Pi(a,b)$.

But even with this description, it's not clear whether $\Pi(a,b)$ is non zero or admissible.

Actually, ~~what~~ the key result to get $\underline{\text{Th III.4}}$ is

$\underline{\text{Th}}$ III.6 (BB, after an idea of Colmez)

There is an isomorphism of B-reps:

$$\widehat{\pi(a,b)}^* \simeq \left(\varprojlim_{\psi} D(a,b)\right)^b$$

where:

$B = $ Borel, $D(a,b)$ (φ,Γ)-module $/\mathcal{E}$ attached to $V_{a,b}$

(crypt. rep st $D_{cris}(V_{a,b}) = D_{cris}(a,b)$),

$$\left(\varprojlim_{\psi} D(a,b)\right)^b = \{(x_n) \in D(a,b)^{\mathbb{N}},\ \forall n\ \psi(x_{n+1}) = x_n,$$
$$\{x_n, n\in\mathbb{N}\} \text{ bounded for the weak top}\}$$

Action of B:

* $\begin{pmatrix} a & \\ & 1\end{pmatrix} \cdot (x_n)_n = (\sigma_a(x_n))_n \quad a \in \mathbb{Z}_p^*$
* $\begin{pmatrix} p & \\ & 1\end{pmatrix} \cdot (x_n)_n = (x_{n+1})_n$ (shift)
* $\begin{pmatrix} 1 & b \\ & 1\end{pmatrix} \cdot (x_n)_n = \left((1+T)^{p^n b} x_n\right)_{n>>0}$. (action of $\begin{pmatrix} 1 & 1 \\ & 1\end{pmatrix}$ = mult by $1+T$)

$\underline{\text{Rk}}$: $\left(\left(\varprojlim_{\psi} D(a,b)\right)^b\right)^* = \mathrm{Hom}^c_{\mathcal{O}_L}\left(\left(\varprojlim_{\psi} D^0(a,b)\right)^b, L\right)$ endowed with the top sup norm is a unitary Banach rep of B (because $\left(\varprojlim_{\psi} D^0(a,b)\right)^b$ = compact)

Sketch of proof of $\underline{\text{Th III.6}}$:

We'll construct a map: $\left(\varprojlim_{\psi} D(a,b)\right)^b \longrightarrow \widehat{\pi(a,b)}^*$

$\underline{\text{Fact}}$ (Wach, Kedlaya, Berger): $\left(\varprojlim_{\psi} D(a,b)\right)^b = \left(\varprojlim_{\psi} M(a,b)\right)^b$

with $M(a,b) = \{f = f_a \otimes e_a + f_b \otimes e_b \in \mathcal{R} \otimes D_{cris},$
$f_a \in \mathcal{R}^+$ of order $v_p(a)$
$f_b \in \mathcal{R}^+$ —— $v_p(b)$
$\varphi^{-n}(f) \in \mathrm{Fil}^0(L_n[[t]] \otimes D_{cris}(a,b))\ \forall n \geq 0\}$

$\underline{\text{th. II.11}}$: $M(a,b) = (\mathcal{R}^+ \otimes D_{cris}(a,b)) \cap D^{\dagger}$

Inside $\pi(a,b)$ have the B-stable subspace of functions with compact support, call it $\pi(a,b)_c$. [defined by exchanging α,β in the parab induction, not the $LL(D_{pst} \cdots)$]

Have an iso $I: \pi(b,a) \xrightarrow{\sim} \pi(a,b)$ (here we use $a \neq b$, $p^{\pm 1} b$)

$$f(z) \otimes P(z) \longmapsto P(z) \otimes \int_{\mathcal{O}_p} \sigma(x+z) f(x)\, dx$$

dx: Haar

But: $I(\pi(b,a)_c) \neq \pi(a,b)_c$. Actually, can check that

$$\pi(a,b) = \pi(a,b)_c + I(\pi(b,a)_c).$$

So to get an element of $\pi(a,b)^*$, need to give 2 linear forms μ_a, μ_b on $\pi(a,b)_c$ & $\pi(b,a)_c$, s.t.:

$$\int \phi_b \mu_b = \int I(\phi_b) \mu_a \qquad \forall \phi_b \in \pi(b,a)_c$$
$$\text{s.t. } I(\phi_b) \in \pi(a,b)_c$$

(compatibility condition)

Let $(f_n) \in \left(\varprojlim_{\psi} M(a,b)\right)^b$, with

$$f_n = f_{a,n} \otimes e_a + f_{b,n} \otimes e_b \qquad \forall n$$

(i) the sq is bounded $\Leftrightarrow$ $(f_{a,n})_n$ bdd in $\mathcal{R}^+_{v_p(a)}$ (idem b)

(ii) ψ-compatibility:

$$\psi(f_{n+1}) = \psi(f_{a,n+1} \otimes a\varphi(e_a)) + \psi(f_{b,n+1} \otimes b\varphi(e_b))$$
$$= a\,\psi(f_{a,n+1}) \otimes e_a + b\psi(f_{b,n+1}) \otimes e_b$$

So ψ-comp $\Leftrightarrow$ $(a^n f_{a,n}) \in \varprojlim_{\psi} \mathcal{R}^+_{v_p(a)}$ (idem b)

Amice-Fourier: $\{\text{distributions on } \mathbb{Z}_p\} \simeq \mathcal{R}^+$

$$\mu \mapsto A_\mu = \int_{\mathbb{Z}_p} (1+T)^x \mu(x)$$

$$\psi(A_\mu) = \int_{p\mathbb{Z}_p} (1+T)^{x/p} \mu(x).$$

$$\Rightarrow \quad (LA_c(\mathbb{Q}_p, L))^* \xrightarrow{\sim} \varprojlim_{\psi} \mathcal{R}^+$$
$$\mu \underset{\text{B-eq}}{\longmapsto} \left(\mathrm{Res}_{\mathbb{Z}_p} \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \mu\right)_{n \geq 0}$$

So $(a^n f_{a,n})_n$ can be seen as a linear form μ'_a on $\pi(a,b)_c$ (integrate against μ_a), idem for b.

A <u>VERY NICE</u> computation ([BB], Lem. 5.1.1, 5.1.2) shows that the condition $\varphi^{-n}(f_{a,n} \otimes e_a + f_{b,n} \otimes e_b) \in \mathrm{Fil}^0(L_n[[t]] \otimes D_{cris}(a,b))$ $\forall n \geq 1$ $\Leftrightarrow$ $\int \phi_b \mu'_b = C(a,b) \int I(\phi_b) \mu'_a$ for all ϕ_b ....

with $C(a,b) = \dfrac{1 - b/pa}{1 - a/b}$. (!)

let $\mu_a = \mu'_a$, $\mu_b = C(a,b)^{-1}\mu'_b$.

we see that (μ_a, μ_b) defines an element of $\pi(a,b)^*$.

Hence, have: $\left(\varprojlim_{\psi} D(a,b)\right)^{b} \longrightarrow \pi(a,b)^*$.

Have to show that it lands in $\widehat{\pi(a,b)}^* \subset \pi(a,b)^*$.

By def,

$$\widehat{\pi(a,b)}^* = \{\mu \in \pi(a,b)^*,\ \mu(\mathcal{L}) \text{ bounded in } L\}$$

with $\mathcal{L} = \mathcal{O}_L[G]\phi$, for any $\underset{\neq 0}{\phi} \in \pi(a,b)$

(lattice because $\pi(a,b)$ irred $\Rightarrow \pi(a,b) = L[G]\phi$)

$$= \{\mu \in \pi(a,b)^*,\ \mu(g\phi) \text{ bounded indep of } g \in G\}$$

$$\underset{\substack{\text{Iwasawa dec}\\ G = BK\\ \text{compact}}}{=} \{\mu \in \pi(a,b)^*,\ \mu(g\phi) \text{ bdd indep of } g \in B\}$$

(Rk: Once again, a priori not clear that there exists such a $\mu \neq 0$!)

If $g \in B$, $\mu = (\mu_a, \mu_b)$ as before, $\phi = \phi_a + I(\phi_b)$

$$\int (g\cdot\phi)\mu = \int (g\phi_a)\mu_a + \int (g\phi_b)\mu_b$$

Recall: Amice-Vélu-Vishik

$r \in \mathbb{Q}$, $d > r-1$. $\mu \in \mathcal{C}^r(\mathbb{Z}_p, L)^* \Leftrightarrow \exists C_\mu \in L\ \forall \lambda \in \mathbb{Z}_p, \forall j \in \{0, \dots, d\}$

$\mu \in \mathrm{Locpol}^d(\mathbb{Z}_p, L)^*$

$$\int_{\lambda + p^n\mathbb{Z}_p} (z-\lambda)^j \mu(z) \in C_\mu p^{n(j-r)}\mathcal{O}_L.$$

Here $d = k-2$, $v_p(a) + v_p(b) = k-1$

$v_p(a), v_p(b) > 0$ hence $v_p(a), v_p(b) < d+1$.

$g = \begin{pmatrix} 1 & -\lambda \\ 0 & p^n \end{pmatrix}$

$\phi = \mathbb{1}_{\mathbb{Z}_p} z^j$

$$\int_{\mathbb{Q}_p} g\phi\,\mu = p^{-nj}\int_{\lambda + p^n\mathbb{Z}_p} (z-\lambda)^j \mu(z) \in C_\mu p^{-nr}\mathcal{O}_L$$

$\Rightarrow$ bounded.

Alternative argument: Let Π be the dual of $(\varprojlim_{\psi} D(a,b))^{b}$

As $(\varprojlim_{\psi} D(a,b))^{b} = L \otimes \mathcal{O}_L$ compact, it's a unitary admissible Banach rep of B. Moreover $\widehat{\pi(a,b)}$ is the completion of $\pi(a,b)$ wrt a $\mathcal{O}_L[B]$-module generating of f.t. (Iwasawa dec again)

$\Rightarrow$ the map $\pi(a,b) \to \Pi$ gives $\widehat{\pi(a,b)} \to \Pi$ by def

-- (à revoir!)

Conversely it's not difficult to see that any element of $\widehat{\pi(a,b)}^{*}$ comes from an element of $(\varprojlim_{\psi} D(a,b))^{b}$. This concludes the sketch of pf of Th III.6. $\square$

Rk: We have seen a miracle in the course of the pf: the existence of the intertwining operator $I: \pi(b,a) \simeq \pi(a,b)$ on the G-side is related to the Hodge filtration on the Galois side!

Cor III.7: $\widehat{\pi(a,b)} \neq 0$.

Pf: Indeed, $(\varprojlim_{\psi} D(a,b))^{b}$ is $\neq 0$, because it contains $D^{\psi=1}$, which is $\neq 0$, because (Cherbonnier-Colmez, Iwasawa theory) $D^{\psi=1}$ generates D as an $\mathcal{E}$-vs. $\square$

Rks: (a) The method does not apply when $a=b$. But the result is still true in this case (Paskunas).

(b) If $a=pb$, $\pi(a,b) \simeq \mathrm{Sym}^{k-2}(L^2) \otimes_L \underbrace{\mathrm{Ind}_B^G |\cdot| \otimes |\cdot|^{-1}}_{\text{ext of } 1 \text{ by St}}$

In this case (as far as I understand...), the good object is still the universal unitary completion of $\pi(a,b)$, which is $\simeq B(a,b)/L(a,b)$, and to $B(b,a)/\left(\overline{L(b,a) + \text{space } v_p(--)}\right)$ (not $B(b,a)/L(b,a)$!)

see Lect IV

("$v_p = \log_\infty$")