LECT IV: p-ADIC LLC: THE FUNCTOR $V \mapsto \Pi(V)$ AND ITS PROPERTIES

We saw last time that if V is a 2-dim'l crystalline rep of $\mathcal{G}_{\mathbb{Q}_p}$, the UUC of $LL(WD(D_{cris}(V))) \otimes \mathrm{Sym}^{k-2}(L^2)$ (HT wts of $V = 0, k-1$) is a good candidate for $\Pi(V)$. This does not work at all for other de Rham representations, even the semi-stable (non crystalline) ones: in this case, the smooth rep is always a twist of the Steinberg rep, and Breuil has shown that the UUC of $St \otimes \mathrm{Sym}^{k-2}(L^2)$ is <u>never</u> admissible for $k > 2$. Moreover, there exist (uncountably) many admissible filtrations on $D_{st}(V)$ in general, so the rep we look at should rather be quotient of the UUC, depending in a subtle way on the Hodge filtration.

Nevertheless, we also saw that (notation of Lecture III, Th III.6):

$$\widehat{\pi(a,b)}^* \simeq \left(\varprojlim_{\psi} D(a,b)\right)^b$$

and it was Colmez' wonderful idea that the RHS could still be the right object for general Galois reps (even non de Rham ones!), even if the LHS is not.

<u>Goal of today</u>: state the main properties of the p-adic LLC, using (φ,Γ)-modules.

Let V be a (2-dim'l ← not necessarily) p-adic rep of $\mathcal{G}_{\mathbb{Q}_p}$, $D \in \Phi\Gamma^{ét}(\mathcal{E})$ the associated étale (φ,Γ)-module. On D, one has an action of the <u>mirabolic</u> (a part of it) $P^+ = \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ (mirabolic $P = \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$) by:

$$\begin{pmatrix} p^k a & b \\ & 1 \end{pmatrix} \cdot x = (1+T)^b \varphi^k(\sigma_a(x)).$$

Let $D \boxtimes \mathbb{Q}_p := \varprojlim_{\psi} D$ and $(D \boxtimes \mathbb{Q}_p)_b := (\varprojlim_{\psi} D)^b$ — on these spaces, have an action of P (same formulas as last week)

& even an action of B, if we choose a unitary character δ and require the center to act via δ.

|| $D \boxtimes \mathbb{Q}_p$ is actually the space of glob. sect of a B-eq sheaf on $\mathbb{Q}_p$, $U \mapsto D \boxtimes U$ st $D \boxtimes (a + p^n\mathbb{Z}_p) = (1+T)^a \varphi^n(D)$, $a \in \mathbb{Z}_p$, $n \geq 0$.

Unfortunately, we cannot extend this to an action of G in general (i.e. define the action of $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$). The case $D = \mathcal{R}$ is instructive:

$$(\mathcal{E} \boxtimes \mathbb{Q}_p)^b = (\mathcal{E}^+ \boxtimes \mathbb{Q}_p)^b = \text{bdd measures on } \mathbb{Q}_p$$
$$= \text{dual of (continuous f}^{\text{ons}} \text{ on } \mathbb{Q}_p \text{ going to } 0 \text{ at } \infty)$$

(whereas $\mathcal{E}^+ \boxtimes \mathbb{Q}_p$ = measures on $\mathbb{Q}_p$ = dual of (continuous f$^{\text{ons}}$ on $\mathbb{Q}_p$ with compact support))

with the action of B given by the dual of

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} f\right)(x) = \delta(a)\,\delta(ad)^{-1} f\left(\frac{dx-b}{a}\right)$$
$$= \delta(d)^{-1} f\left(\frac{dx-b}{a}\right).$$

We see that the good space is rather the dual of

$$B(\delta) := \{ C^0\ f : \mathbb{Q}_p \to L,\quad x \mapsto \delta(x)\,\phi(\tfrac{1}{x})\ C^0 \text{ at } 0 \}$$

with $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(x) = \delta(ad-bc)^{-1}\,\delta(a-cx)\, f\left(\frac{dx-b}{a-cx}\right)$.

As $\mathbb{P}^1(\mathbb{Q}_p)$ is obtained by gluing two copies of $\mathbb{Z}_p$ along $\mathbb{Z}_p^*$ via $x \mapsto \frac{1}{x}$, the map $\mu \mapsto (\mathrm{Res}_{\mathbb{Z}_p} \mu_1, \mathrm{Res}_{\mathbb{Z}_p} w_\delta \mu)$ induces an iso between $B(\delta)^*$ and

$$\mathcal{D}_0(\mathbb{Z}_p, L) \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) := \{ (\mu_1, \mu_2) \in \mathcal{D}_0(\mathbb{Z}_p, L)^{\oplus 2},\ \mathrm{Res}_{\mathbb{Z}_p^*} \mu_1 = w_\delta(\mathrm{Res}_{\mathbb{Z}_p^*} \mu_2) \}$$

where $w_\delta :$ ~~measures on $\mathbb{Z}$~~ $\mathcal{D}_0(\mathbb{Z}_p^*, L) \to \mathcal{D}_0(\mathbb{Z}_p^*, L)$ is the involution defined by:

$$\int_{\mathbb{Z}_p^*} \phi\, w_\delta(\mu) = \int_{\mathbb{Z}_p^*} \delta(x)\, \phi\left(\frac{1}{x}\right) \mu.$$

We want to translate this in terms of (φ, Γ)-modules: i.e. we want to define an involution w_δ on $\mathcal{E}^+$ st $w_\delta(A_\mu) = A_{w_\delta(\mu)}$. Using Riemann sums, we find that

$$w_\delta(z) = \lim_{n \to \infty} \sum_{i \in \mathbb{Z}_p^*/p^n} \delta(1/i)(1+T)^i\, \sigma_{-i^2}(\varphi^n \psi^n((1+T)^{-i^{-1}} z))$$

Here is what Colmez shows: Let $D \in \Phi\Gamma^{et}(\mathcal{E})$. For any $z \in D \boxtimes \mathbb{Z}_p^*$, the series

$$\lim_{n \to \infty} \sum_{i \in \mathbb{Z}_p^*/p^n} \delta(i^{-1})(1+T)^i \sigma_{-i^2}(\varphi^n(\psi^n((1+T)^{-i^{-1}} z)))$$

converges to an element $w_\delta(z)$ in $D \boxtimes \mathbb{Z}_p^*$.

Let $D \boxtimes \mathbb{P}^1(\mathbb{Q}_p) = \{ (z_1, z_2) \in D^2,\ \mathrm{Res}_{\mathbb{Z}_p^*}(z_1) = w_\delta(\mathrm{Res}_{\mathbb{Z}_p^*}(z_2)) \}$

and if $z = (z_1, z_2)$ let:

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z = (z_2, z_1)$
- $a \in \mathbb{Q}_p^*$ $\quad \begin{pmatrix} a & \\ & a \end{pmatrix} \cdot z = (\delta(a) z_1, \delta(a) z_2)$
- $a \in \mathbb{Z}_p^*$ $\quad \begin{pmatrix} a & \\ & 1 \end{pmatrix} \cdot z = \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} z_1, \delta(a) \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} z_2 \right)$
- $z' = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot z$, then $\mathrm{Res}_{p\mathbb{Z}_p} z' = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z_1$, $\mathrm{Res}_{\mathbb{Z}_p} w z' = \delta(p) \psi(z_2)$
- $b \in p\mathbb{Z}_p$, $z' = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} z$, then

  $\mathrm{Res}_{\mathbb{Z}_p} z' = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} z_1 \qquad \mathrm{Res}_{p\mathbb{Z}_p} w z' = u_b(\mathrm{Res}_{p\mathbb{Z}_p}(z_2))$

  with $u_b = \delta^{-1}(1+b) \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} \circ w_\delta \circ \begin{pmatrix} (1+b)^2 & b(1+b) \\ & 1 \end{pmatrix} \circ w_\delta \circ \begin{pmatrix} 1 & \frac{1}{1+b} \\ & 1 \end{pmatrix}$

  on $D \boxtimes p\mathbb{Z}_p$

<u>Th</u> IV.1 (Colmez) Given D and δ as above, $\exists !$ extension of the B-eq sheaf $U \mapsto D \boxtimes_\delta U$ on $\mathbb{Q}_p$ to a G-eq sheaf (again denoted $(U \mapsto D \boxtimes_\delta U)$) on $\mathbb{P}^1(\mathbb{Q}_p)$, with the action of G on $D \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ given by the above formulas.

<u>Rk</u>. This works in any dimension!

This is only a first step: we need to introduce a quotient of $D \boxtimes \mathbb{P}^1$ to get sthg reasonable. In the example of $B(\delta)$ before, we had $B(\delta)^* \simeq \mathcal{E}^+ \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$, not $\mathcal{E} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$. What is the analogue of $\mathcal{E}^+$ for a (φ, Γ)-module?

<u>Def</u> IV.2 A <u>treillis</u> in a (φ, Γ)-module D over $\mathcal{O}_\mathcal{E}$ is a sub $\mathcal{O}_\mathcal{E}^+$-module <u>compact</u> st its image in $D/\pi_L D$ is a lattice of this $k_\mathcal{E}$-vs. If D is an étale (φ, Γ)-module over $\mathcal{E}$, a treillis of D is a treillis of a (φ, Γ)-stable $\mathcal{O}_\mathcal{E}$-lattice in D.

<u>Th</u> IV.3: (Herr-Colmez) Let $D \in \Phi\Gamma^{\acute{e}t}(\mathcal{E})$. The set of all treillis M s.t. $\psi(M) = M$ has a smallest element called $D^\natural$.

<u>Def</u> IV.4: Let $D^\natural \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) = \{ z \in D \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p), \mathrm{Res}_{\mathbb{Q}_p}(z) \in D^\natural \boxtimes \mathbb{Q}_p \}$

it's a B-stable subspace. (stable by Γ by unicity, by ψ by def, not by φ)

Rk: It's not the set of global sections of a sheaf on $\mathbb{P}^1(\mathbb{Q}_p)$!

Def IV.5: Say (D,δ) is OK (G-compatible) if $D^\natural \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ is G-($\Leftrightarrow$ w-) stable in $D \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$. If so, set

$$\Pi_\delta(D) = D \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) / D^\natural \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p).$$

Th IV.6 (Colmez) given (D,δ) OK, we have:

(i) $\Pi_\delta(D) \in \mathrm{Ban}(G)$, of finite length.

(ii) If $\check{D}$ is the (φ,Γ)-module associated to $V^* \otimes \chi$, $(\check{D},\delta^{-1})$ is ok, have an exact sq of G-rep:

$$0 \to \Pi_{\delta^{-1}}(\check{D})^* \to D \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) \to \Pi_\delta(D) \to 0$$

(in other words, $\Pi_\delta(D)^* \simeq \check{D}^\natural \boxtimes_{\delta^{-1}} \mathbb{P}^1(\mathbb{Q}_p)$)

Rk: a) The last exact sequence is a generalization of the exact sequence:

$$0 \to \mathcal{D}_0(\mathbb{P}^1(\mathbb{Q}_p),L) \to \mathcal{E} \boxtimes \mathbb{P}^1(\mathbb{Q}_p) \to \mathcal{C}^0(\mathbb{P}^1(\mathbb{Q}_p),L) \to 0$$

(take $D = \mathcal{E}$, $\delta = 1$) b) ~~Given (i), (ii) is not difficult.~~

Thm IV.7 (Colmez, Paskunas, Dospinescu) Fix $\delta : \mathbb{Q}_p^\times \to O_L^\times$. Set
$\mathcal{C}(\delta) = \{ D \in \Phi\Gamma^{\acute{e}t}(\mathcal{E}), (D,\delta) \text{ is OK} \}$. Then:

(i) $\mathcal{C}(\delta) \to \mathrm{Ban}(G)(\delta)^{fl} /$ finite dim'l rep is an equivalence of cat.
$D \mapsto \Pi_\delta(D)$

(ii) $\mathcal{C}(\delta)$ contains all 1-dim'l (φ,Γ)-modules

(iii) $\mathcal{C}(\delta)$ does not contain any abs. irred D of dim ≥ 3.

(iv) Say $D \in \Phi\Gamma^{\acute{e}t}(\mathcal{E})$ is abs. irred of dim 2. Then:
$D \in \mathcal{C}(\delta) \iff \det D = \delta \cdot \chi$

Rk: a) (ii) was already seen: $\mathcal{E}^+ \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ is $B(\delta)^* = (\mathrm{Ind}_B^G(\delta^{-1} \otimes 1)^{cont})^*$

and $\mathcal{E} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) \to \mathrm{Ind}_B^G(\chi^{-1}\delta \otimes \chi^{-1})^{cont}$ def by
$z \mapsto \mathrm{res}_0\left(\mathrm{Res}_{\mathbb{Z}_p}(wgz)\frac{dT}{1+T}\right)$ induces an iso $\Pi_\delta(\mathcal{E}) \simeq B(\chi\delta^{-1})$.

(b) (Dospinescu's thesis, lem. 5.5.2)

let (D,δ) be G-compatible. $\mathrm{Res}_{\mathbb{Q}_p} : z \mapsto (\mathrm{Res}_{\mathbb{Z}_p}(\mathbb{P}^n{}_1) z)$

$D \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) \to D \boxtimes_\delta \mathbb{Q}_p$

induces an exact sq:

$$0 \to (0, D^{nr}) \to (D^{G} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p))_{ns} \to (D^{G} \boxtimes_\delta \mathbb{Q}_p)_b \to 0$$

$= \bigcap_n \varphi^n(D)$; $(D^{G} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p))_{ns} = \{z \in D^G \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p),\ \mathrm{Res}_{\mathbb{Q}_p}(z) \in D^G \boxtimes \mathbb{Q}_p\}$

$= 0$ if D irred of dim ≥ 2 ; $= D^G \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$

(the example of $\mathcal{E}$ in dim 1 shows that the first term is $\neq 0$ in dim 1)

Hence for $\dim D = 2$ abs irred, $D^G \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) \simeq (D^G \boxtimes_\delta \mathbb{Q}_p)^b$

which explains the connection with Berger-Breuil.

(c) Given (ii) of Thm. IV.6 & rk (b) before, (i) of Thm IV.6 is not difficult: one shows that if M is a closed $\mathcal{O}_L$-submodule of $D_0^G \boxtimes_\delta \mathbb{Q}_p$ (D_0 over $\mathcal{O}_\mathcal{E}$), stable by P, st $\mathrm{Res}_{\mathbb{Z}_p}(M)$ generates D_0 as a (φ,Γ)-module, $D_0^G \boxtimes_\delta \mathbb{Q}_p \subset M$. This shows that $(\check{D}^G \boxtimes_{\delta^{-1}} \mathbb{Q}_p)_b$ ~~and hence also $(\check{D}^G \boxtimes_{\delta^{-1}} \mathbb{Q}_p)_b$ is~~ is irreducible as a topological B-module ($= (\check{D}_0^G \boxtimes_\delta \mathbb{Q}_p) \otimes_{\mathcal{O}_L} L$). As by IV.6 (ii) + rk (b), $\Pi_\delta(D,\delta)^* \simeq_{B\text{-eq}} (\check{D}^G \boxtimes_{\delta^{-1}} \mathbb{Q}_p)_b$, this gives that $\Pi_\delta(D)$ is irreducible as a B-rep (!). Then do the 1-dim'l case by hand, and induction.

<u>Assume from now on</u>: $\dim D = 2$, $\delta = \chi^{-1} \det(D)$. Set $\Pi(D) = \Pi_\delta(D)$, $\check{\Pi}(D) = \Pi_{\delta^{-1}}(\check{D})$.

Then $\Pi_{\delta^{-1}}(\check{D}) = \Pi_\delta(D) \otimes \delta^{-1}$, i.e. $\check{\Pi}(D) = \Pi(D) \otimes \delta^{-1}$

$$\rightsquigarrow\ 0 \to \Pi(D)^* \otimes \delta^{-1} \to D \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p) \to \Pi(D) \to 0$$

<u>Rk:</u> In particular, $\mathrm{Ext}^1_G(\Pi(D), \Pi(D)^* \otimes \delta^{-1}) \neq 0$. Is this a general phenomenon?

Thm IV.8 (Colmez, Paškūnas, Dospinescu)
(cf lect I)
The ~~map~~ functor $V \mapsto \Pi(D(V))$ induces a bijection

$$\left\{ \begin{array}{l} \text{abs irred 2-dim'l} \\ C^0 \text{ rep of } \mathcal{G}_{\mathbb{Q}_p} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{abs irred} \\ \text{non ordinary} \end{array} \Pi \in \mathrm{Ban}(G) \right\}$$

(non ordinary: subquotient of the parabolic (continuous) induction of a unitary character of the torus)

Rk. One can classify : 2-dim'l rep of $\mathcal{G}_{\mathbb{Q}_p}$ on k_L-vs & smooth adm. rep. of G ~~…~~ on a k_L-vs. } semi-simple
(Local Langlands mod p)
and define an explicit correspondence between these objects (Breuil).

If V is as in Th IV.8, T an $\mathcal{O}_L$-stable lattice in V and $\bar{V}$ its reduction mod π_L, $\bar{V}$ does not depend on the choice on T.

In the same, if $\Pi \in \mathrm{Ban}(G)$, and Θ is the unit ball of some G-inv norm on Π inducing its top, $\bar{\Pi} := \Theta \otimes_{\mathcal{O}_L} k_L$ does not depend on the choice on the norm. One can check that $\overline{\Pi(D(V))}$ corresponds to $\bar{V}$ by LL mod p (i.e. p-adic LL is compatible with mod p LL !).

The proof of Thm IV.7 (iv) when $\dim D = 2$ uses a p-adic continuation argument : one first shows the stability of $D^\psi \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ ~~…~~ for a Zariski-dense subset of Galois rep in the deformation space of a fixed residual rep and show that all the constructions behave well in family. This subset is ~~…~~ given by <u>trianguline rep</u> :

Def IV.9 (Colmez) A rep V of $\mathcal{G}_{\mathbb{Q}_p}$ is <u>trianguline</u> if $D_{rig}(V)$ is a successive ext of rk1 (φ,Γ)-modules over $\mathcal{R}$.

Recall that every (φ,Γ)-module over $\mathcal{R}$ of rk 1 is of the form $\mathcal{R}(\delta)$, $\delta : \mathbb{Q}_p^\times \to L^\times$ (étale iff δ unitary). Def : $w(\delta) = \frac{\log_p \delta(u)}{\log_p(u)}$ (u et top $\mathbb{Z}_p$)

$u(\delta) = \mathrm{val}_p(\delta(p))$

(weight & slope of δ)

If V is 2 dim'l trianguline, $\exists\ \delta_1, \delta_2$ st

$$0 \to \mathcal{R}(\delta_1) \to D_{rig}(V) \to \mathcal{R}(\delta_2) \to 0$$

but δ_1, δ_2 are not assumed to be unitary, hence $D_{rig}(V)$ (and V) can be irreducible (this is what makes the notion of triang rep interesting!).

As $D_{rig}(V)$ is étale, one has $u(\delta_1)+u(\delta_2)=0$ & $w(\delta_1), w(\delta_2)$ are the HT wts of V.

Let $S = \{ (\delta_1, \delta_2, \mathcal{L}),\ \delta_1, \delta_2 : \mathbb{Q}_p^\times \to L^\times,\ \mathcal{L} = \infty$ if $\delta_1\delta_2^{-1} \neq \begin{cases} x^i & i \geq 0 \\ |x|x^i & i \geq 1 \end{cases}$

$\mathcal{L} \in \mathbb{P}^1(L)$ otherwise $\}$

$\forall s \in S$, $\exists$ $D(s)$ triangular (not étale) (φ,Γ)-module over $\mathcal{R}$ ext of $\mathcal{R}(\delta_2)$ by $\mathcal{R}(\delta_1)$.

Let $S_{irr} = S_*^{cris} \sqcup S_*^{st} \sqcup S_*^{ng}$, with

$$S_*^{cris} = \{ s \in S,\ u(\delta_1)+u(\delta_2)=0,\ w(\delta_1)-w(\delta_2) \in \mathbb{Z}_{\geq 1},\ \mathcal{L}=\infty,\ w(\delta_1)-w(\delta_2) > u(\delta_1) > 0 \}$$

$$S_*^{st} = \{ s \in S,\ u(\delta_1)+u(\delta_2)=0,\ w(\delta_1)-w(\delta_2) \in \mathbb{Z}_{\geq 1},\ \mathcal{L} \in \mathbb{P}^1(L) \setminus \{\infty\},\ w(\delta_1)-w(\delta_2) > u(\delta_1) > 0 \}$$

$$S_*^{ng} = \{ s \in S,\ u(\delta_1)+u(\delta_2)=0,\ u(\delta_1)>0,\ w(\delta_1)-w(\delta_2) \notin \mathbb{Z}_{\geq 1} \}.$$

<u>Th IV.10</u>: If $s \in S_{irr}$, $D(s)$ is étale and $V(s)$ is trianguline irreducible. Conversely, every 2-dim'l irred trianguline rep is of the form $V(s)$ (after possibly extending L), with $s \in S_{irr}$.

If $s \in S_*^{cris}$, $V(s)$ is crystalline over an abelian ext of $\mathbb{Q}_p$, after possibly twisting by a character.

If $s \in S_*^{st}$, $V(s)$ is semi-stable ($\overline{\text{abelian ext useless}}$).

If $s \in S_*^{ng}$, $V(s)$ is not a twist of a de Rham rep.

Every rep V which becomes semi-stable over an abelian ext of $\mathbb{Q}_p$ is trianguline.

<u>Rk</u>: If $s \in S_*^{cris} \sqcup S_*^{st}$, $s = (\delta_1, \delta_2, \mathcal{L})$, one can check that $\mathcal{L} \in \mathbb{P}^1(L)$ is the [illegible] such that the non trivial step of the Hodge filtration on $D_{dR}(V(s)) \otimes_L L_\infty$ is given by $L_\infty(e_1 - \mathcal{L} e_2)$

(case $\mathcal{L}=\infty$ different....)

(cryst. case) αx^{-a}
$\delta_1 = \beta x^b$ $a<b$ HT.
$\delta_2 = \alpha x^a$ βx^{-b}

For irred trianguline reps, you can, as in the crystalline case (lect III) describe $D^{\natural} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ $(\delta = \chi^{-1} \det D)$ and check that it is stable by G:

let $\Pi(s) = \Pi(V(s))$, $\log_{\mathcal{L}}$ the logarithm normalized by $\log_{\mathcal{L}}(p) = \mathcal{L}$ (if $\mathcal{L} = \infty$, set $\log_\infty = v_p$), $\delta_s = \delta_1 \delta_2^{-1} (x|x|)^{-1}$. If $s \in S_{irr}$, the only case where $\mathcal{L} \neq \infty$ is when $\delta_1 \delta_2^{-1} = x^i$, $i \geq 0$ (because assumed $u(\delta_1) > 0$)

- let $B(s) = \{ f : \mathbb{Q}_p \to L$, class $\mathcal{C}^{u(\delta_1)}$, $x \mapsto \delta_s(x) f(\frac{1}{x})$ extends by continuity to a class $\mathcal{C}^{u(\delta_1)}$ function $\}$

Action of G:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f \right)(x) = \delta_2(ad-bc)\, \delta_s(a-cx)\, f\left(\frac{dx+b}{a-cx}\right)$$

~~(ad-bc)|ad-bc| δ_1^{-1}(ad-bc) δ_s(...)~~

(hope I got the formula right...)

Rk: As said before in the crystalline case, $\delta_2 = x^a \alpha$, $\delta_1 = x^b \beta$ $\quad a < b$ HT wts

to find back [BB] formulas.

- let $M(s)$ = closure of the space ~~generated by~~:
  - * if $\delta_s \neq x^i$, $i \geq 0$, generated by 1 and $x \mapsto \delta_s(x - \lambda)$, $\lambda \in \mathbb{Q}_p$.
  - * if $\delta_s = x^i$, $\cap$ of $B(s)$ & the space generated by $x \mapsto \delta_s(x - \lambda)$ and $x \mapsto \delta_s(x-\lambda) \log_{\mathcal{L}}(x - \lambda)$, $\lambda \in \mathbb{Q}_p$.

- ~~Let then~~ $\Pi(s) \simeq (B(s) / \hat{M}(s))^* \otimes \delta$ (Colmez $s \in S_*^{st} \sqcup S_*^{ng}$, Berger-Breuil $s \in S_*^{cris}$)

Then (Thm IV.11) $D^{\natural} \boxtimes_\delta \mathbb{P}^1(\mathbb{Q}_p)$ ___

Rk: If $s \in S_*^{st}$, where do these linear combinations of $\log_{\mathcal{L}}$ come from?

It comes from the fact that you have to add $\log_{\mathcal{L}} T$ to the Robba ring to describe $D(s)$ as did [BB] in the cryst. case, in terms of p-adic functional analysis: $D(s)^{\dagger} \otimes \mathcal{R}[\frac{1}{t}, \log_{\mathcal{L}} T] \simeq D_{cris}(s) \otimes \mathcal{R}[\frac{1}{t}, \log_{\mathcal{L}} T]$.