An arithmetic lattice of $\mathrm{Isom}(\mathbb{H}^d)$

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A lattice in $\mathbb{R}^n$ is a discrete $\mathbb{Z}$-submodule of $\mathbb{R}^n$ of rank n. The group $\mathrm{GL}(n,\mathbb{R})$ acts transitively on the space of lattices, and the stabilizer of the standard lattice $\mathbb{Z}^n$ is the discrete subgroup $\mathrm{GL}(n,\mathbb{Z})$. The space $\mathcal{L}_n$ of lattices in $\mathbb{R}^n$ thus identifies witht the orbifold quotient $\mathrm{GL}(n,\mathbb{R})/\mathrm{GL}(n,\mathbb{Z})$. This endows $\mathcal{L}_n$ with a natural topology.

We will admit the following compactness criterion, due to Mahler:

Theorem 0.1. *A subset A of $\mathcal{L}_n$ has compact closure if and only if there exist constants $\varepsilon > 0$ and R such that for every lattice Λ in A we have*

$$\mathbf{Vol}(\mathbb{R}^n/\Lambda) \leq R$$

and

$$\Lambda \cap B(0,\varepsilon) = \{0\} \ .$$

Let q be the quadratic form on $\mathbb{R}^n$ given by

$$q(x) = x_1^2 + \ldots + x_{n-1}^2 - \sqrt{2}x_n^2 \ .$$

Define $\Gamma = \mathrm{O}(q) \cap \mathrm{GL}(n, \mathbb{Z}[\sqrt{2}])$. The goal of this exercise is to prove that Γ is a uniform lattice in the group $\mathrm{O}(q)$ of linear transformations preserving q. Since q has signature $(n-1,1)$, Γ is a uniform lattice in $\mathrm{Isom}(n-1,1)$.

Let $\overline{q}$ denote the quadratic form

$$\overline{q}(x) = x_1^2 + \ldots + x_{n-1}^2 + \sqrt{2}x_n^2 \ ,$$

Image of q by the Galois automorphism of $\mathbb{Q}[\sqrt{2}]$. Let Q and Q' be the quadratic forms on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ given respectively by

$$Q(u,v) = q(u + \sqrt{2}v) + \overline{q}(u - \sqrt{2}v)$$

and

$$Q'(u,v) = \frac{1}{\sqrt{2}}\left(q(u+\sqrt{2}v) - \overline{q}(u-\sqrt{2}v)\right) \ .$$

1. Show that Q and Q' take integral values on $\mathbb{Z}^n \times \mathbb{Z}^n$.

Let G be the subgroup of $\mathrm{GL}(\mathbb{R}^{2n})$ preserving Q and Q'. Let Λ_0 denote the lattice $\mathbb{Z}^n \times \mathbb{Z}^n$ in $\mathbb{R}^{2n}$.

2. Assume that there exists a sequence $(u_n) \in \Lambda_0$ and a sequence $(g_n) \in G$ such that $g_n \cdot u_n \underset{n\to+\infty}{\longrightarrow} 0$. Show that for n large enough, $Q(u_n) = Q'(u_n) = 0$.

3. Show that $u_n = 0$ for n large enough. Deduce that the G-orbit of Λ_0 is relatively compact in $\mathcal{L}_{2n}$.

Let (g_n) be a sequence in G such that $g_n \cdot \Lambda_0$ converges to a lattice Λ.

4. Show the existence of $h_n \in \mathrm{GL}(2n, \mathbb{Z})$ such that $g_n h_n$ converges to some $g \in \mathrm{GL}(2n, \mathbb{R})$.

5. For every $u \in \Lambda_0$, show that

$$Q(h_n \cdot u) = Q(g \cdot u)$$

for n large enough.

6. Deduce that the G-orbit of Λ_0 is closed in $\mathcal{L}_n$.

Define

$$\begin{array}{rccc} \varphi : & \Gamma & \to & \mathrm{GL}(\mathbb{Z}^n \times \mathbb{Z}^n) \\ & A + \sqrt{2}B & \mapsto & \begin{pmatrix} A & 2B \\ B & A \end{pmatrix} \end{array} .$$

7. Show that φ is an injective group morphism and that

$$\varphi(\Gamma) = G \cap \mathrm{GL}(\mathbb{Z}^n \times \mathbb{Z}^n) \ .$$

8. Show that there exists an isomorphism $\psi : G \to \mathrm{O}(q) \times \mathrm{O}(\overline{q})$ such that

$$\pi \circ \psi \circ \varphi = i \ ,$$

where $\pi : \mathrm{O}(q) \times \mathrm{O}(\overline{q}) \to \mathrm{O}(q)$ denotes the projection on the first factor and $i : \Gamma \to \mathrm{O}(q)$ denotes the inclusion.

9. Conclude.