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Compact quotients of reductive homogeneous spaces and Surface group representations

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Introduction

Ce mémoire a pour objet de présenter une synthèse de ma recherche, depuis les travaux issus de ma thèse commencée en 2011 jusqu'à ses développements plus récents et ses perspectives futures. L'exercice n'est pas aisé en raison de la relative diversité des thèmes que j'ai abordé et de la variété des outils mathématiques impliqués. Dans cette introduction je m'efforcerai de mettre en avant la cohérence de l'ensemble de mes travaux, en commençant par un aperçu historique de ce qu'on appelle la *géométrisation des variétés*, puis en présentant les ramifications de ma recherche à partir de cette problématique initiale.

Géométriser les variétés

Les variétés topologiques ou différentielles sont par essence des objets flexibles, définis « à déformation près ». Pour mieux les décrire, on est souvent amené à les *géométriser*, c'est-à-dire (en un sens très large) à les munir de structures géométriques qui reflètent par certains aspects leurs propriétés topologiques. On peut donner plusieurs acceptions à cette notion vague de géométrisation. Ici nous nous préoccuperons principalement de munir nos variétés de *structures localement homogènes* au sens d'Ehresmann, c'est-à-dire de les identifier localement à un certain espace homogène. Une telle structure fait appaître naturellement une action du groupe fondamental de la variété sur cet espace homogène, ce qui lie étroitement l'étude des structures localement homogènes sur une variété à celle des représentations linéaires de son groupe fondamental.

De l'uniformisation des surfaces de Riemann à l'invention du groupe fondamental

Historiquement, la géométrisation des variétés est indissociable de l'invention du groupe fondamental, et tous deux puisent leur origine dans l'uniformisation des surfaces de Riemann compactes.

L'essor de la théorie des fonctions analytiques de la variable complexe a conduit les mathématiciens du XIX$^{\text{e}}$ siècle à accepter la notion de fonction multivaluée (sur un domaine du plan complexe ou plus généralement sur une surface de Riemann). Une telle fonction possède localement plusieurs *branches*, qui sont permutées par le prolongement analytique le long d'un chemin fermé. Lorsque la fonction y est solution de certaines équations différentielles, les transformations qui permutent les branches sont parfois bien identifiées. Par exemple, les branches d'une solution de l'équation

$$y' = f$$

sont permutées par des translations, tandis que celles d'une solution de l'équation

$$\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = f$$

sont permutées par des homographies. [1]

En étudiant ces fonctions multivaluées, on comprend progressivement qu'elles permettent d'*uniformiser* les courbes algébriques complexes, c'est-à-dire que leur réciproque est parfois définie sur un domaine simplement connexe et identifie la surface de Riemann de départ au quotient de ce domaine par le groupe de permutation des branches. Ainsi, Eisenstein, Liouville et Weierstrass comprennent que les intégrales elliptiques permettent d'identifier les courbes elliptiques aux quotients de $\mathbb{C}$ par des réseaux, puis Klein et Poincaré démontrent que les courbes algébriques de genre supérieur à 2 sont des quotients du plan hyperbolique par un *groupe fuchsien.* [2]

Le théorème d'uniformisation est présent dans l'esprit de Poincaré lorsque, quelques années plus tard, il invente la notion de groupe fondamental dans l'*Analysis Situs* [157]. Poincaré y introduit informellement le groupe fondamental d'une variété comme le groupe des permutations des branches d'une fonction multivaluée « la plus générale possible » (autrement dit, le groupe d'automorphismes de son revêtement universel). Il en donne pour exemple les suspensions de difféorphismes du tore, qu'il décrit explicitement comme des quotients de $\mathbb{R}^3$ par un groupe de transformations affines, et dont il écrit :

> L'analogie avec la théorie des groupes fuchsiens est trop évidente pour qu'il soit nécessaire d'insister.

Les variétés qui ont motivé l'introduction du groupe fondamental sont donc géométriques : leur revêtement universel est un espace homogène, et leur groupe fondamental un groupe discret de transformations de cet espace.

Les espaces localement homogènes

Géométriser une variété, en un sens plus précis, consisterait donc à « incarner » son revêtement universel et son groupe fondamental dans une géo-

1. Les spécialistes reconnaîtront dans le terme de gauche la *dérivée schwarzienne* de la fonction y.

2. Pour plus de précisions sur l'histoire du théorème d'uniformisation, on pourra consulter [54].

métrie au sens de Klein, c'est-à-dire un espace homogène. Cette idée a été formalisée dans un cadre général par Ehresmann [59].

Remarquant que certaines structures géométriques sur les variétés différentielles (une métrique de courbure constante, un connection plate...) fournissent des identifications locales avec certains espaces homogènes, Ehresmann introduit la notion générale de variété localement modelée sur un espace G-homogène X, c'est-à-dire munie d'identifications locales avec X qui sont bien définies modulo une transformation de G. Les identifications locales se prolongent alors analytiquement en une application multivaluée de notre variété vers l'espace X, dont les branches sont permutées par des transformations de G. En termes modernes, une variété M localement modelée sur X est munie d'une *application développante* $\mathbf{dev} : \widetilde{M} \to X$ qui est équivariante par rapport à une représentation $\mathbf{hol} : \pi_1(M) \to G$ appelée *holonomie*. Ehresmann étudie alors la complétude de ces espaces localement homogènes. Sous certaines conditions (en particulier lorsqu'M est compacte et X un espace homogène riemannien), l'application développante est un difféomorphisme global et la variété M est donc un quotient de X.

Si l'étude des espaces localement homogènes jusque dans les années 70 est surtout marquée par de puissants théorèmes de rigidité (Calabi–Weil, Mostow, Margulis...), à la fin des années 70, Thurston reprend les travaux d'Ehresmann et démontre un théorème très général de déformation : si $\mathbf{hol} : \pi_1(M) \to G$ est l'holonomie d'une structure localement modelée sur un espace G-homogène X, alors tout morphisme suffisemment proche de $\mathbf{hol}$ est l'holonomie d'une structure localement homogène proche de la structure initiale. Ce *principe d'Ehresmann–Thurston*[3] et l'utilisation qu'en fera Thurston dans ses travaux de géométrisation des 3-variétés conduisent à un regain d'intérêt pour la géométrisation des variétés, intérêt encore accru au cours de la dernière décennie avec la découverte des *représentations Anosov*, source de nombreux nouveaux exemples de géométrisation.

Avant de préciser un peu cette notion, je voudrais présenter quelques problèmes de géométrisation spécifiques qui ont contribués à l'essor de ce domaine de recherche.

Groupes kleinéens et hyperbolisation des 3-variétés

Rappelons que le groupe de Lie $\mathrm{PSL}(2, \mathbb{C})$ est le groupe des isométries directes de l'espace hyperbolique $\mathbb{H}^3$. Il agit par homographies sur la sphère de Riemann, qui s'identifie au bord à l'infini de $\mathbb{H}^3$.

Les *groupes kleinéens* sont les sous-groupes discrets de $\mathrm{PSL}(2, \mathbb{C})$, et leur classification revient donc essentiellement à classifier les 3-variétés hyperboliques complètes. Une sous-classe importante des groupes kleinéens, stable par petites déformations, est celle des *groupes kleinéens convexe-cocompacts*,

3. Cette terminologie est due à Bergeron et Gelander [24], d'après qui le théorème peut être lu entre les lignes des travaux d'Ehresmann [60].

qui sont les groupes fondamentaux de 3-variétés hyperboliques compactes à bord convexe.

Si les premiers exemples de groupes kleinéens apparaissent dans les travaux de Schottky et Klein en lien avec l'uniformisation des surfaces de Riemann, leur étude systématique commence en 1960 avec les travaux d'Ahlfors et Bers [3]. En résolvant l'équation de Beltrami sous des hypothèses de régularité très faibles, ces derniers démontrent que les déformations d'un groupe kleinéen convexe-cocompact sont paramétrées par les structures conformes sur le bord de la variété hyperbolique associée.

Une quinzaine d'année plus tard, Thurston énonce sa conjecture de géométrisation des 3-variétés, qui prédit que toute 3-variété compacte sans bord asphérique et atoroïdale possède une structure hyperbolique. En s'appuyant sur les travaux d'Ahlfors et Bers, il démontre cette conjecture pour les variétés *Haken*, qui peuvent être découpées le long de surfaces incompressibles [195]. En simplifiant beaucoup, sa preuve consiste à hyperboliser les variétés à bord obtenues en découpant le long d'une surface incompressible, puis à déformer convenablement les structures hyperboliques sur chaque morceau (grâce au théorème d'Ahlfors–Bers) de façon à pouvoir les recoller. La preuve nécessite une compréhension fine des représentations fidèles et discrètes de groupes de surfaces, pour laquelle Thurston développe considérablement la théorie de Teichmüller.

Ces résultats conduisent Thurston à formuler un certain nombre de grandes conjectures – outre la conjecture d'hyperbolisation, on peut citer la conjecture virtuellement Haken et la conjecture des laminations terminales – dont la résolution (respectivement par Perelman [156], Agol [2], et Brock–Canary–Minsky [33]) a abouti à une compréhension profonde de la topologie et géométrie des variétés de dimension 3.

Convexes divisibles

Au XIX$^{\text{e}}$, l'essor de la géométrie hyperbolique et de la géométrie projective remettent en cause le caractère absolu de la géométrie euclidienne et conduisent Klein, dans son célèbre *programme d'Erlangen* [104], à définir la géométrie comme l'étude des espaces homogènes. Lorsqu'Ehresmann introduit ensuite la notion de variété localement homogène, il est alors conduit naturellement à s'intéresser plus spécifiquement aux variétés localement modelées un l'espace projectif.

Une sous-classe intéressante de telles variétés est formée par les quotients compacts d'ouverts convexes. Les convexes possédant de telles actions cocompactes sont appelés *convexes divisibles*. Cette notion a été introduite par Kuiper dans [115], et étudiée plus avant par Benzécri dans sa thèse [22]. Benzécri pense que les convexes divisibles sont des objets rigides, et démontre effectivement un théorème de rigidité sous une hypothèse de régularité : les convexes divisibles à bord $\mathcal{C}^2$ sont projectivement équivalents au modèle de

Klein de l'espace hyperbolique. Mais quelques années plus tard, Katz et Vinberg construisent des convexes divisibles dont le bord est $\mathcal{C}^1$ mais pas $\mathcal{C}^2$ [100], tandis que Koszul démontre un principe d'Ehresmann–Thurston pour les convexes divisibles [114] : si $\Gamma \subset \mathrm{PSL}(n, \mathbb{R})$ divise un convexe de $\mathbb{R}\mathbf{P}^n$, les petites déformations de Γ continuent à diviser un convexe. [4] On peut en particulier obtenir des familles continues de convexes divisibles en déformant un réseau uniforme de $\mathrm{SO}(d,1) \simeq \mathrm{Isom}(\mathbb{H}^d)$ à l'intérieur de $\mathrm{SL}(d+1, \mathbb{R})$.

Enfin, Choi et Goldman [43] (en dimension 2) puis Benoist [19] (en toute dimension) démontrent que le fait de diviser un convexe est aussi une condition fermée. Ainsi, toute déformation continue d'un réseau hyperbolique uniforme de $\mathrm{SO}(d,1) \simeq \mathrm{Isom}(\mathbb{H}^d)$ dans $\mathrm{SL}(d+1, \mathbb{R})$ divise un convexe. Le théorème de Choi–Goldman aboutit en un certain sens à une classification des convexes divisibles de dimension 2. En revanche, les espaces de déformation des convexes divisbles de dimension 3 et leurs liens avec la géométrisation des 3-variétés restent encore incompris, malgré les travaux de Benoist [20] et, plus récemment, de Ballas–Danciger–Lee [12].

Espaces-temps de courbure constante

Une autre motivation de l'étude des espaces localement homogènes prend sa source dans la théorie de la relativité. Aux alentours de 1900, Lorentz, Poincaré et Einstein comprennent progressivement que le groupe des symétries des lois de la physique n'est pas le groupe des transformations « galliléennes » mais le *groupe de Lorentz* $\mathrm{SO}(3,1)$, qui préserve les équations de Maxwell. Quelques années plus tard, Einstein intègre les relations entre gravitation et accélération dans la courbure de l'espace-temps : selon la théorie de la relativité générale, l'espace-temps est une variété lorentzienne dont la courbure est reliée aux *tenseur énergie impulsion* par l'équation d'Einstein.

Si on lui postule une certaine homogénéité [5], l'espace-temps devrait ressembler à grande échelle à une variété lorentzienne de courbure constante. Le signe de cette courbure, qui dépend de la valeur de la constante cosmologique et conditionne les propriétés d'expansion de l'univers, a fait l'objet de plusieurs controverses dans l'histoire de la relativité générale. L'étude de la forme de l'univers soulève donc un problème de géométrisation : quelles sont les variétés localement modelées sur un espace homogène lorentzien de courbure constante ?

Les mathématiciens s'emparent de ce problème à la fin des années 50 tout en s'éloignant un peu des préoccupations physiques qui l'ont motivé.

4. N'ayant pas connaissance du principe d'Ehresmann–Thurston, Koszul démontre en fait seulement que les structures projectives convexes forment un ouvert de toutes les structures projectives.

5. C'est le *principe cohomologique parfait*, introduit par Bondi en 1950 et dont la pertinence peut-être discutée, ce pour quoi je ne suis pas compétent et que je m'abstiendrai donc de faire.

Markus, avec Calabi [39] et Auslander [11], initie notamment l'étude des variétés lorentziennes compactes de courbure constante, qui connaîtra de nombreux développements sur lesquels nous reviendrons en détail dans le chapitre 2.

Cette hypothèse de compacité est toutefois peu plausible d'un point de vue physique car elle implique une « récurrence temporelle » qui viole le principe de causalité. Une hypothèse plus réaliste est celle d'un espace-temps *Globalement Hyperbolique Cauchy compact* (GHC), c'est-à-dire, très schématiquement, d'un espace-temps compact dans les directions d'espace mais qui vérifie une condition de causalité.

Dans un article remarquable [145], Mess décrit les variétés lorentziennes GHC de courbure constante en dimension 3, et exhibe les liens étroits entre leurs espaces de déformations et l'espace de Teichmüller de leur surface de type espace. Certains de ses résultats seront étendus par la suite à la dimension supérieure. Dans le cas de courbure négative, en particulier, les variétés lorentziennes GHC de dimension $d+1$ sont (sous une hypothèse supplémentaire de convexité) des quotients d'un ouvert de l'*espace anti-de Sitter* AdS^{d+1} par le groupe fondamental d'une variété compacte de courbure négative de dimension d. Barbot démontre dans [13] que toute déformation continue de ce groupe dans le groupe de isométries de l'espace anti-de Sitter est encore le groupe fondamental d'un espace-temps anti-de Sitter GHC.

Le paradigme des groupes Anosov

La notion de *sous-groupe Anosov* d'un groupe de Lie semisimple G ou de *représentation Anosov* d'un groupe de type fini Γ à valeurs dans G a été introduite par Labourie au tournant du siècle [118], et rapidement développée par Guichard et Wienhard [84] ainsi que plusieurs autres auteurs. Pour reprendre une expression d'Anna Wienhard, la théorie des représentations Anosov est à l'origine d'un « changement de paradigme » dans l'étude des sous-groupes discrets des groupes de Lie : alors que la géométrie au XX$^{\mathrm{e}}$ siècle est marqué par les théorèmes de rigidité (de Mostow et Margulis notamment), au milieu desquels les rares familles de groupes discrets non rigides (les groupes kleinéens, les groupes de symétries de convexes divisibles...) semblent quelques contre-exemples disparates, le formalisme des groupes Anosov a transformé cette vision des choses en révélant toute la richesse des déformations de groupes discrets et en fournissant un cadre général à leur étude.

Précisons un peu. Considérons un groupe de Lie semisimple réel G (par exemple $\mathrm{SO}(p,q)$ ou $\mathrm{SL}(n,\mathbb{R})$). Les espaces G-homogènes compacts sont essentiellement les *variétés de drapeaux*[6] G/P, où P est un *sous-groupe parabolique*. Disons qu'une suite $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ est P-proximale si elle possède un

6. Lorsque $G = \mathrm{SL}(n,\mathbb{R})$, ce sont effectivement les ensembles de drapeaux de $\mathbb{R}^n$ d'un certain type.

point fixe attractif dans G/P. Informellement, un sous-groupe discret Γ de G est P-Anosov s'il vérifie une propriété de P-proximalité uniforme.

Cette propriété (énoncée précisément au chapitre 1, Section 1.3.2) a de nombreuses conséquences géométriques et dynamiques remarquables. Elle implique que le groupe Γ est *hyperbolique* au sens de Gromov [95], et que son bord à l'infini s'identifie à un fermé Γ-invariant de G/P. Guichard et Wienhard montrent également qu'un groupe P-Anosov Γ agit proprement discontinument et cocompactement sur un ouvert d'une autre variété de drapeaux G/Q [84], enrichissant ainsi considérablement les exemples de variétés compactes localement modelées sur des variétés de drapeaux.

La principale vertu de la propriété Anosov est sa stabilité structurelle : les petites déformations d'un groupe P-Anosov restent P-Anosov. Lorsque G est le groupe des isométries d'un espace hyperbolique (ou plus généralement lorsque G est de rang 1), son unique variété de drapeaux est le bord à l'infini de l'espace hyperbolique, et la propriété Anosov est alors équivalente à la propriété de convexe-cocompacité. La principale source d'exemples de groupes Anosov consiste alors à déformer ces sous-groupes convexe-cocompacts dans des groupes de Lie de rang supérieur. Ainsi, les déformations de réseaux uniformes de $\mathrm{SO}(d,1)$ dans $\mathrm{SL}(d+1)$ ou $\mathrm{SO}(d,2)$ (et, plus généralement, les groupes qui divisent un convexe strict d'un espace projectif et les groupes fondamentaux d'espaces-temps anti-de Sitter GHC convexes) sont des exemples de groupes Anosov. Récemment, Zimmer [209] et Danciger–Guéritaud–Kassel [52] on montré que la propriété Anosov pouvait toujours se ramener à une propriété de *convexe-cocompacité projective* introduite par Crampon–Marquis [50], qui synthétise la théorie des groupes convexe-cocompacts en rang 1 et celle des convexes divisibles.

Outre le fait qu'elle fournit un cadre unifié à divers exemples de structures géométriques flexibles, le plus grand succès de la théorie des groupes Anosov est la description des propriétés géométriques de certaines familles de représentations de groupes de surfaces. Rappelons qu'une surface compacte Σ de genre supérieur à 2 possède des structures hyperboliques, et que l'holonomie de chaque structure hyperbolique fournit une représentation injective d'image discrète de son groupe fondamental Γ dans $\mathrm{PSL}(2,\mathbb{R})$ appelée *représentation fuchsienne*. Les représentations fuchsiennes modulo conjugaison forment une composante connexe de la *variété des caractères* $\mathfrak{X}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ qui s'identifie d'après le théorème d'uniformisation à l'*espace de Teichmüller* de Σ. Dans [118], Labourie montre que les *représentations de Hitchin*, qui forment une composante connexe de la variété des caractères $\mathfrak{X}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$, sont toutes Anosov. Avec Burger, Iozzi et Wienhard, ils démontrent ensuite le même résultat pour les *représentations maximales* à valeurs dans des groupes hermitiens [35]. Ces exemples donnent naissance à la *théorie de Teichmüller supérieure*, qui étudie les composantes de représentations Anosov dans les variétés de caractères de groupes de surfaces et leurs similarités avec l'espace de Teichmüller. Une part importante de ma recherche se situe dans ce domaine.

Ramifications d'un parcours de recherche

Mes travaux de recherche trouvent leur motivation initiale dans des problèmes de géométrisation, et les représentations Anosov y jouent un rôle prépondérant. Dans la suite de ce mémoire, j'ai choisi d'organiser mes travaux selon deux thèmes traités de façon à peu près indépendante : les quotients compacts d'espaces homogènes réductifs d'une part, et les représentations de groupes de surfaces d'autre part.

Dans cette introduction, je propose de raconter mon parcours de recherche d'une façon plus chronologique que logique, afin d'illustrer comment ma problématique initiale (l'étude des variétés pseudo-riemanniennes compactes localement homogènes) a ramifié dans plusieurs directions autonomes. Chaque section ci-dessous présente une de ces branches.

Espaces pseudo-riemanniens localement symétriques

J'ai commencé ma thèse en m'intéressant aux variétés compactes localement modelées sur des espaces pseudo-riemanniens symétriques. Une question centrale dans leur étude est celle de leur *complétude* : ces espaces sont-ils nécessairement des quotients de leur modèle symétrique ? J'abordai cette question dans le cas des variétés localement modelées sur un groupe de Lie de rang 1 muni de sa métrique de Killing, un choix motivé entre autres par le fait que les quotients compacts de ces espaces étaient en passe d'être très bien compris grâce aux les travaux en cours de Guéritaud–Guichard–Kassel–Wienhard [81, 80], et qu'il est frustrant de ne pas savoir si l'on décrit ainsi toutes les variétés compactes localement modelées sur ces espaces. J'obtins le théorème suivant :

Théorème 1. *Soit G un groupe de Lie de rang 1 muni de l'action de $G \times G$ par multiplication à gauche et à droite, et U un ouvert de G. Supposons qu'il existe un sous-groupe de $G \times G$ agissant proprement discontinûment et cocompactement sur U. Alors $U = G$.*

Ce résultat très partiel a néanmoins le mérite d'impliquer que les quotients compacts de G ne peuvent pas être déformés continûment en des structures incomplètes.

Après m'être un peu cassé les dents sur la question de la complétude, je décidai de m'intéresser à la seule géométrie de rang 1 où elle est résolue : le cas de $G = \mathrm{PSL}(2,\mathbb{R})$, qui s'identifie à l'espace anti-de Sitter AdS^3. Les quotients compacts de $\mathrm{PSL}(2,\mathbb{R})$ par un sous-groupe de $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ sont décrit très précisément par les travaux de Kulkarni–Raymond [117] et Kassel [96] : ils sont (à revêtement fini près) de la forme

$$j \times \rho(\Gamma)\backslash\mathrm{PSL}(2,\mathbb{R}) \ ,$$

où Γ est le groupe fondamental d'une surface compacte de genre supérieur à 2, j est une représentation fuchsienne et ρ une représentation *dominée par* j au sens où il existe $f : \mathbb{H}^2 \to \mathbb{H}^2$ qui est (j, ρ)-équivariante et contractante. Il restait à décrire l'ensemble de ces couples (j, ρ). Notamment, est-ce que toute représentation non-fuchsienne $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ peut être dominée par une représentation fuchsienne ?

Cette question intéressait Bertrand Deroin par son lien avec une autre question ouverte : quelles représentations $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ sont l'holonomie d'une *structure hyperbolique branchée* ? Pour aborder ces deux questions, nous commençâmes par développer une approche basée sur des applications harmoniques discrètes, qui permet de voir ρ comme l'holonomie d'une structure hyperbolique "pliée" avec des singularités coniques d'angles supérieurs à 2π. Cette approche (récemment menée au bout par Florestan Martin–Baillon [142]) fournit rapidement une réponse positive à la première question si l'on sait traiter un certain nombre de triangulations dégénérées. Mais nous comprîmes au bout d'un moment qu'au prix d'un peu d'analyse, ces difficultés techniques disparaissent si l'on remplace les applications harmoniques discrètes par de véritables applications harmoniques. En outre, on contrôle ainsi mieux les paramètres de la construction : chaque choix de structure complexe sur Σ fournit une représentation fuchsienne qui domine ρ, et je démontrai enfin qu'on obtient ainsi toutes les représentations qui dominent ρ.

Théorème 2 (Deroin–Tholozan [55, 189])**.** *Soit ρ une représentation non fuchsienne de Γ dans* $\mathrm{PSL}(2, \mathbb{R})$ *(ou même dans n'importe quel groupe de Lie de rang* 1*). Alors l'ensemble des représentations fuchsiennes qui dominent ρ modulo conjugaison est non vide et homéomorphe à l'espace de Teichmüller de Σ.*

Ce théorème permet donc de décrire l'*espace des modules* des variétés anti-de Sitter compactes de dimension 3.

Pour conclure mon étude de ces variétés, je cherchais à calculer leur volume. Grâce aux travaux de Guéritaud–Kassel [81], on sait que ces variétés fibrent en géodésiques de type temps au dessus d'une surface compacte. En intégrant le long des fibres, j'obtins l'expression suivante pour leur volume :

Théorème 3 (Tholozan [191])**.** *Soit Γ un groupe de surface, $j : \Gamma \to$* $\mathrm{PSL}(2, \mathbb{R})$ *une représentation fuchsienne et $\rho : \Gamma \to$* $\mathrm{PSL}(2, \mathbb{R})$ *une représentation dominée par j. Alors*

$$\mathbf{Vol}\left(j \times \rho(\Gamma) \backslash \mathrm{PSL}(2, \mathbb{R})\right) = \frac{\pi^2}{2} \left|\mathbf{eu}(j) + \mathbf{eu}(\rho)\right| ,$$

où **eu** *désigne la classe d'Euler.*

Ce résultat s'étend aisément aux quotients de $\mathrm{SO}(d, 1)$, toujours en s'appuyant sur leur structure de fibré fournie par Guéritaud–Kassel.

Le fait surprenant que le volume d'une variété de dimension 3 soit calculé par une « classe caractéristique » appelait une explication plus conceptuelle qu'un simple calcul. Je finis par comprendre qu'on peut contourner l'utilisation des travaux de Guéritaud–Kassel en raisonnant « à homotopie près » ce qui me permis de généraliser le théorème précédent de la façon suivante :

Théorème 4 (Tholozan [191])**.** *Soit* $\Gamma\backslash G/H$ *le quotient compact d'un espace homogène réductif. Alors*

$$\mathbf{Vol}(\Gamma\backslash G/H) = \int_{[\Gamma]} i^*\omega_{G/H}$$

où $\omega_{G/H}$ *est une classe de cohomologie continue de* G *dépendant uniquement de* H*,* i *est l'inclusion de* Γ *dans* G *et* $[\Gamma]$ *est une « classe fondamentale » dans l'homologie de* Γ*.*

Je déterminai également à quelle condition la classe $\omega_{G/H}$ est une classe caractéristique (impliquant la rationalité du volume) et déduisis de ce théorème une formule explicite pour le volume des quotients de $\mathrm{SU}(d,1)$. Surtout, je trouvai plusieurs critères d'annulation de $\omega_{G/H}$, qui fournissent une puissante obstruction à l'existence de quotients compacts de certains espaces homogènes réductifs.

Représentations de groupes de surfaces et applications harmoniques

Le théorème de domination des représentations de groupes de surface en rang 1 obtenu avec Bertrand Deroin ouvrait une nouvelle perspective de recherche : nous nous sommes naturellement demandé ce qu'il advenait de ce résultat pour des représentations en rang supérieur, à commencer par les représentations de Hitchin dans $\mathrm{PSL}(3,\mathbb{R})$.

Soit donc Σ une surface compacte de genre supérieur à 2 et Γ son groupe fondamental. Chaque représentation de Hitchin $\rho : \Gamma \to \mathrm{PSL}(3,\mathbb{R})$ agit proprement discontinûment et cocompactement sur un ouvert convexe Ω_ρ du plan projectif en préservant sa *métrique de Hilbert*, dont les propriétés métriques capturent les propriétés algébro-géométriques de ρ. Des résultats de Crampon [49] et Nie [153] sur l'*entropie* de ces convexes montre qu'on ne peut pas espérer dominer uniformément ces représentations par des représentations fuchsiennes. En m'intéressant au sujet, j'appris l'existence d'une métrique riemannienne naturelle sur chaque convexe, la *métrique de Blaschke*, issue de la théorie des sphères affines. Cette métrique est uniformément comparable à la métrique de Hilbert et de courbure supérieure à -1, ce qui suggère qu'à l'inverse du rang 1, les représentations de Hitchin dominent toujours une représentation fuchsienne. Pour affiner ce résultat, il manquait une comparaison précise entre la métrique de Blaschke et celle de Hilbert, ce que j'obtins dans [190].

Théorème 5 (Tholozan [190]). *Soit ρ une représentation de Hitchin d'un groupe de surface Γ à valeurs dans* $\mathrm{PSL}(3,\mathbb{R})$. *Il existe une représentation fuchsienne j telle que*

$$L_\rho(\gamma) \geq L_j(\gamma)$$

pour tout $\gamma \in \Gamma$.

Ici, $L_j(\gamma)$ désigne la longueur de translation de $j(\gamma)$ dans le plan hyperbolique et $L_\rho(\gamma)$ cette de $\rho(\gamma)$ dans Ω_ρ muni de sa métrique de Hilbert. Ce nouveau résultat de comparaison fine entre représentations d'un groupe de surface utilise encore une fois des outils d'analyse harmonique. La métrique de Blaschke de Ω_ρ est en effet liée à l'unique application harmonique conforme ρ-invariante de $\widetilde{\Sigma}$ à valeurs dans l'espace symétrique $\mathrm{PSL}(3,\mathbb{R})/\mathrm{PSO}(3)$.

Encouragé dans l'idée que les applications harmoniques tordues peuvent fournir des informations précises sur les représentations de groupes de surfaces, je m'intéressai ensuite aux représentations maximales à valeurs dans le groupe de Lie hermitien $\mathrm{SO}(2,d)$. Il devenait alors difficile de contourner la théorie des *fibrés de Higgs*, ces objets holomorphes qui capturent les propriétés algébriques des applications harmoniques tordues à valeurs dans les espaces symétriques. En étudiant en détail la structure de ces fibrés de Higgs, nous montrâmes avec Brian Collier et Jérémy Toulisse que si $\rho : \Gamma \to \mathrm{SO}(2,d)$ est une représentation maximale, toute application harmonique conforme ρ-equivariante à valeurs dans l'espace symétrique de $\mathrm{SO}(2,d)$ est l'application de Gauss d'une *surface maximale de type espace* dans l'espace pseudo-riemannien $\mathbb{H}^{2,d-1}$. En donnant à cette surface maximale le rôle de la sphère affine pour les représentations de Hitchin dans $\mathrm{PSL}(3,\mathbb{R})$, nous obtînmes de nombreuses propriétés géométriques de ces représentations :

Théorème 6 (Collier–Tholozan–Toulisse [45]). *Soit $\rho : \Gamma \to \mathrm{SO}(2,d)$ une représentation maximale. Alors :*

- *Il existe une unique application harmonique conforme ρ-equivariante de $\widetilde{\Sigma}$ dans l'espace symétrique de $\mathrm{SO}(2,d)$, et cette application est un plongement.*
- *Il existe une représentation fuchsienne j telle que $L_\rho \geq L_j$.*
- *Le domaine de discontinuité Ω_ρ de Guichard–Wienhard dans l'espace des plans isotropes de $\mathbb{R}^{2,d}$ admet une fibration ρ-équivariante sur $\widetilde{\Sigma}$ dont les fibres sont des sous-variétés de drapeaux.*

(Ici, $L_\rho(\gamma)$ désigne le logarithme du rayon spectral de $\rho(\gamma)$.)

Ces propriétés diverses et leurs importance dans la théorie de Teichmüller supérieure sont discutées plus amplement dans la section 3.2.

Théorie de Teichmüller suprême

La principale raison pour laquelle les applications harmoniques permettent de décrire aussi finement les représentations de Hitchin et les représentations maximales en rang 2 tient à une coincidence : les fibrés de Higgs associés aux applications harmoniques conformes sont alors *cycliques*, une propriété de structure très forte qui simplifie beaucoup l'étude de ces applications harmoniques. À partir du rang 3, les fibrés de Higgs cycliques ne paramètrent qu'une sous-variété de la variété des caractères, et il semble difficile d'extraire de l'analyse harmonique des propriétés géométriques précises sur toutes les représentations.

Pour poursuivre l'étude des représentations Anosov de groupes de surfaces au-delà du rang 2, il semble alors préférable de les approcher avec les outils de dynamique hyperbolique. Schématiquement, cette approche – qu'on peut attribuer en premier lieu à Sambarino [171] – consiste à associer à une représentation Anosov ρ un *spectre des longueurs* $L_\rho : [\Gamma] \to \mathbb{R}_+$ (où $[\Gamma]$ désigne les classes de conjugaisons d'éléments de Γ) qu'on peut voir comme la fonction des périodes d'une reparamétrisation höldérienne du flot géodésique de Σ. Les travaux de Bowen, Margulis ou Ruelle sur la dynamique des flots d'Anosov fournissent alors des informations précises sur le comportement asymptotique de L_ρ. L'*entropie topologique* de ces flots, qui mesure le taux de croissance exponentielle de L_ρ, joue ici un rôle prépondérant.

Si $\rho : \Gamma \to \mathrm{PSL}(d, \mathbb{R})$ est une représentation de Hitchin, on peut par exemple lui associer un flot dont les périodes sont données par

$$L_\rho(\gamma) = \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)) \ ,$$

où λ_i désigne le logarithme de la i-ème valeur propre [32]. Bertrand Deroin me fit remarquer un jour que l'entropie d'un telle flot doit être égale à 1 puisque que L_ρ se lit sur le cocycle des dérivées d'une action action de Γ de classe $\mathcal{C}^1$ sur le cercle (voir aussi [158]). En essayant de préciser cette remarque, je compris qu'il existe une correspondance bi-univoque entre les trois espaces suivants :

(1) L'espace des reparamétrisations höldériennes du flot géodésique de Σ d'entropie 1, modulo équivalence de Livšic,
(2) L'espace des actions de Γ de classe $\mathcal{C}^{1+Hölder}$ sur le cercle qui sont Hölder conjuguées à une action fuchsienne, modulo conjugaison $\mathcal{C}^1$,
(3) les métriques hyperboliques sur le feuilletage faiblement stable du flot géodésique de Σ, transversalement Hölder, modulo isotopie.

Cette correspondance préserve en outre trois notions naturelles de *fonction des périodes* de $[\Gamma]$ dans $\mathbb{R}_+$.

Je présente cette correspondance dans des notes en cours d'élaboration [192]. Son intérêt est de multiplier les points de vue sur ce que j'aime appeler prétentieusement l'*espace de Teichmüller suprême.*[7] Sa construction n'est que le début d'un programme de recherche qui ambitionne de décrire la

7. Pour ma défense, il existe déjà un « espace de Teichmüller universel » et un « super-espace de Teichmüller ».

géométrie de cet espace de dimension infinie, dans l'espoir d'en déduire des propriétés géométriques des espaces de Teichmüller supérieurs. Le point de vue (3) permet en effet de réintroduire la théorie de Teichmüller classique dans la théorie de Teichmüller supérieure, tandis que le point de vue (2) fournit une analogie entre les représentations Anosov et les représentations maximales dans $\mathrm{Diff}(\mathbb{S}^1)$. Nous donnons plus de détails sur ce projet et ses développements à venir dans la section 3.4.1.

Géométries de Hilbert

L'étude des représentations de Hitchin dans $\mathrm{PSL}(3,\mathbb{R})$ me permis d'approfondir ma compréhension des géométries de Hilbert. En particulier, mon théorème de comparaison entre la métrique de Balschke et la métrique de Hilbert, valable en toute dimension, se trouva répondre à une conjecture de Colbois et Verovic sur la croissance du volume des géométries de Hilbert :

Théorème 7. *Soit Ω un convexe propre de l'espace projectif $\mathbb{R}\mathbf{P}^{d-1}$. Alors l'entropie volumique de Ω :*

$$\mathscr{H}(\Omega) \overset{\mathrm{def}}{=} \limsup_{R\to+\infty} \frac{1}{R}\left(\log \mathbf{Vol}(B(o,R))\right)$$

est inférieure à $d-2$.

($B(o,R)$ désigne ici la boule de centre un point o quelconque de rayon R pour la métrique de Hilbert, et **Vol** désigne son volume pour la mesure de Hausdorff associée.)

Lorsque le convexe Ω est divisé par un groupe Γ, l'entropie volumique coïncide avec l'*exposant critique* de Γ :

$$\delta(\Gamma) = \lim_{R\to+\infty} \frac{1}{R} \log \sharp\{\gamma \in \Gamma \mid d_{Hilb}(o, \gamma\cdot o) \leq R\} .$$

Dans ce cas, le théorème 7 a d'abord été prouvé par Crampon par des méthodes dynamiques [49]. Crampon démontre de plus un théorème de rigidité : si l'entropie est égale à $d-2$, alors le convexe est un ellipsoïde (et sa métrique de Hilbert est la métrique hyperbolique). [8]

Ces résultats invitaient à se demander ce qu'il advient de cet exposant critique lorsque Γ agit seulement convexe-cocompactement sur Ω. L'exposant critique de Γ est alors égal à la croissance du volume du coeur convexe et, lorsqu'Ω est une boule, il est égal à la dimension de Hausdorff de l'ensemble limite de Γ dans $\partial_\infty\Omega$, d'après un célèbre théorème de Sullivan [181]. Daniel Monclair, qui travaillait avec Olivier Glorieux sur ces exposants critiques dans le cas de groupes agissant sur l'espace pseudo-riemannien $\mathbb{H}^{p,q}$ [70],

8. Crampon suppose a priori que Ω est Gromov hyperbolique, mais son résultat a été amélioré par Barthelmé–Marquis–Zimmer [15] en utilisant le théorème 7.

m'expliqua que la théorie de Patterson–Sullivan (et sa généralisation aux espaces Gromov hyperboliques par Coornaert [46]) donne en fait l'égalité entre l'exposant critique de Γ et la dimension de Hausdorff de l'ensemble limite pour la *métrique de Gromov* sur le bord. Il suffisait donc de comparer cette métrique de Gromov à une métrique riemannienne pour obtenir une inégalité sur la dimension de Hausdorff géométrique. Nous avons ainsi obtenu :

Théorème 8 (Glorieux–Monclair–Tholozan [71]). *Soit Γ un groupe agissant de façon convexe-cocompacte sur un ouvert convexe propre et Gromov hyperbolique Ω de l'espace projectif $\mathbf{P}(V)$. Soit Λ_Γ l'ensemble limite de Γ dans $\partial_\infty \Omega$ et posons*

$$\widehat{\Lambda}_\Gamma = \{(x, T_x(\partial_\infty \Omega)), x \in \Lambda_\Gamma\} \subset \mathbf{P}(V) \times \mathbf{P}(V^*) \ .$$

Alors l'exposant critique de Γ pour la métrique de Hilbert est inférieur à la dimension de Hausdorff de $\widehat{\Lambda}_\Gamma$.

Ce théorème est présenté plus en détails dans la section 1.3.3, où nous le replaçons dans le cadre général de l'étude des exposants critiques de groupes Anosov.

Orbites bornées du groupe modulaire

Avec Bertrand Deroin, nous avons continué à nous intéresser aux représentations de groupes de surfaces à valeurs dans $\mathrm{PSL}(2,\mathbb{R})$, et plus particulièrement à la question mentionnée précédemment : Quelles représentations $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ sont les holonomies de *structures hyperboliques branchées* (c'est-à-dire de métriques hyperboliques possédant des singularités coniques d'angles multiples de 2π) ? Les résultats préliminaires que nous avons obtenu (voir Section 3.4.2) nécessitent de découper la surface, d'hyperboliser les morceaux et de les recoller, ce qui conduit naturellement à considérer aussi le problème de géométrisation des surfaces à bord.

Soit $\Sigma_{g,n}$ une surface de genre g avec n composantes de bord et $\Gamma_{g,n}$ son groupe fondamental. La *variété de caractères* $\mathfrak{X}(\Gamma_{g,n}, G)$ est (à peu de choses près) l'espace des représentations de $\Gamma_{g,n}$ dans G modulo conjugaison. Elle est feuilletée par les *variétés de caractères relatives*, où les classes de conjugaison des images des courbes de bord sont fixées, et le *groupe modulaire pur* $\mathrm{MCG}_{g,n}$ de $\Sigma_{g,n}$ (le groupe des classes d'isotopie d'homéomorphismes fixant le bord) agit sur $\mathfrak{X}(\Gamma_{g,n}, G)$ en préservant les variétés de caractères relatives.

En développant une notion de classe d'Euler relative pour les représentations de $\Gamma_{g,n}$ dans $\mathrm{PSL}(2,\mathbb{R})$, nous découvrîmes un peu par hasard l'existence de représentations de $\Gamma_{0,n}$ dont les propriétés surprenantes sont liées au fait qu'elles forment des composantes connexes compactes de certaines variétés de caractères relatives. Au même moment, Gabriele Mondello décrivait la topologie de toutes les variétés de caractères relatives dans $\mathrm{PSL}(2,\mathbb{R})$ en termes de fibrés de Higgs paraboliques [150]. Cela nous encouragea, Jérémy Toulisse et moi, à essayer de construire des variétés de caractères relatives compactes dans des groupes de Lie de rang supérieur en exploitant la correspondance de Hodge non-abélienne parabolique. Nous obtînmes en définitive le théorème suivant :

Théorème 9 (Deroin–Tholozan [56], Tholozan–Toulisse [193])**.** *Soit G l'un des groupes de Lie $\mathrm{PU}(p,q)$, $\mathrm{Sp}(2k,\mathbb{R})$ ou $\mathrm{SO}^*(2k)$. Pour tout $n \geq 4$, il existe un ouvert $\Omega \subset \mathfrak{X}(\Gamma_{0,n}, G)$ qui est la réunion de composantes compactes de variétés de caractères relatives. De plus, les représentations ρ dans Ω ont les propriétés suivantes :*

(1) L'orbite de $[\rho]$ sous l'action de $\mathrm{MCG}_{0,n}$ est bornée,

(2) L'image par ρ de n'importe quelle courbe fermée simple sur $\Sigma_{0,n}$ a toutes ses valeurs propres de module 1,

(3) Pour tout ensemble D de n points dans la sphère de Riemann $\mathbb{S}^2$, l'application harmonique ρ-équivariante de $\widetilde{\mathbb{S}^2 \backslash D}$ dans l'espace symétrique de G est holomorphe.

Ce théorème fournit en particulier de nombreuses familles d'orbites bornées du groupe modulaire et permet de formuler des conjectures sur les propriétés générales de ces orbites bornées. Cela ouvre tout un programme de recherche, dont nous espérons qu'il permettra de comprendre en particulier les orbites finies de $\mathrm{MCG}_{g,n}$, qui sont liées aux représentations linéaire de $\mathrm{MCG}_{g,n+1}$. Ce programme et quelques résultats préliminaires sont présentés dans la section 3.4.3.

Structure du mémoire

Le premier chapitre de ce mémoire est une introduction très générale à l'étude des sous-groupes discrets des groupes de Lie semisimples et leurs déformations. Nous commencerons par quelques rappels sur les groupes de Lie semisimples, leurs espaces symétriques, leurs variétés de drapeaux et l'importance de la projection de Cartan dans leur géométrie (Section 1.1). Nous mentionerons ensuite quelques résultats sur les groupes linéaires et introduirons leurs *variétés de caractères* (Section 1.2). Enfin, nous présenterons les propriétés générales des groupes Anosov et mentionnerons au passage nos résultats sur les géométries de Hilbert (Section 1.3).

Le reste de mes travaux sera présenté au sein de deux chapitres essentiellement indépendants. Le chapitre 2 donne un état des lieux de la recherche sur les quotients compacts d'espaces homogènes réductifs. Nous mentionnerons les diverses obstructions à leur existence (Section 2.2) et les quelques constructions connues de tels quotients (Section 2.3), puis nous discuterons d'une conjecture sur la géométrie de ces quotients, en expliquant notamment comment elle a inspiré mes résultats sur le volume de ces espaces (Section 2.4).

Le chapitre 3, quant à lui, est consacré aux représentations de groupes de surfaces. Après quelques précisions sur leurs variétés de caractères où nous introduirons notamment les espaces de Teichmüller supérieurs (Section 3.1), j'introduirai les applications harmoniques, leurs liens avec les surfaces minimales et les fibrés de Higgs, et j'expliquerai le rôle qu'elles jouent dans mes principaux résultats (Section 3.2). Dans la section 3.3, je présenterai mes constructions de composantes bornées dans les variétés de caractères relatives. Enfin, la section 3.4 esquissera mes projets de recherche en cours autour des représentations de groupes de surfaces : la « théorie de Teichmüller suprême », les métriques hyperboliques branchées et les orbites bornées du groupe modulaire.

Introduction

The purpose of this memoir is to present the results of my research, from my thesis started in 2011 to its more recent and future developments. This is a difficult exercise due to the relative diversity of topics I have worked on and the variety of mathematical tools involved. In this introduction, I will try to put forward the coherence of my works by first giving a historical overview of what we call *geometrizing manifolds*, then describing the ramifications of my research from this initial motivation.

Geometrizing manifolds

Topological or diferentiable manifolds are flexible objects in essence, defined "up to deformations". To understand them better, one is often brought to *geometrize* them, that is (in a very broad sense), to endow them with geometric structures that reflect some of their topological properties. Various significations can be given to this vague notion of geometrization. Here we will be concerned with endowing our manifolds with *locally homogeneous geometric structures* in the sens of Ehresmann, that is, identifying them locally with a certain homogeneous space. Such a structure naturally gives rise to an action of the fundamental group of the manifold on this homogeneous space, so that locally homogeneous structures on a manifold are intimately related to linear representations of their fundamental group.

From the Uniformization of Riemann surfaces to the invention of the fundamental group

Historically, the geometrization of manifolds is inseparable from the invention of the fundamental group, and both take their source in the Uniformization of Riemann surfaces.

The development of the theory of analytic functions of a complex variable lead XIX$^{\text{th}}$ century mathematicians to accept the notion of *multivalued function* (on a complex domain of more generally on a Riemann surface). Such a function locally has several *branches* that are permuted by analytic continuation along a closed path. When the function y is a solution of some differential equation, the transformations permuting the branches are sometimes well-identified. For instance, the branches of a solution of the equation

$$y' = f$$

are permuted by translations, while those of a solution of the equation

$$\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = f$$

are permuted by homographies. [9]

While studying these multivalued functions, mathematicians progressively understood that they can *uniformize* complex algebraic curves, meaning that the inverse function is defined on a simply connected domain and identifies the initial Riemann surface with a quotient of this domain under the group of permutation of the branches. First, Eisenstein, Liouville and Weierstrass understood that elliptic integrals identify the elliptic curves to quotients of $\mathbb{C}$ by lattices, then Klein and Poincaré prove that algebraic curves of higher genus are quotients of the hyperbolic plane by a *Fuchsian group*. [10]

Poincaré has in mind his uniformization theorem when, a few years later, he invents the notion of fundamental group in his *Analysis Situs* [157]. There, he introduces the fundamental group of a manifold informally as the group of permutations of the branches of the "most general multivalued function". (in modern words, the automorphism group of its universal covering). He gives as an example the suspensions of diffeomorphisms of the torus, which he describes explicitely as quotients of $\mathbb{R}^3$ under a group of affine transformations. He even writes:

> The analogy with the theory of Fuchsian groups is too obvious for us to need emphasizing it

We see that the manifolds which motivated the introduction of the fundamental group are geometric: their universal cover is a homogeneous space, and their fundamental group is a discrete group of transformations of this space.

Locally homogeneous spaces

Geometrizing a manifold, in a more precise sense, would thus consist in "incarnating" its universal cover and its fundamental group in a geometry in

9. Experts will have recognized that the left term is the *Schwarzian derivative* of the function y.

10. For more details about the History of the Uniformization theorem, I encourage you to read the book *Uniformization of Riemann surfaces – Revisiting a hundred year old theorem* [54].

the sense of Klein, that is, a homogeneous space. This idea was formalized in a general context by Ehresmann [59].

After pointing out that some geometric structures on differentiable manifolds (constant curvature metrics, flat connections...) provide local identifications with certain homogeneous spaces, Ehresmann introduces the general notion of manifold locally modelled on a G-homogeneous space X, that is, endowed with local identifications with X which are well-defined modulo a transformation of G. The local identifications then extend analytically to a multivalued function from our manifold to the space X, the branches of which are permuted by transformations of G. In modern terms, a manifold M locally modelled on X has a *developing map* $\mathbf{dev} : \widetilde{M} \to X$ which is equivariant with respect to a representation $\mathbf{hol} : \pi_1(M) \to G$ called the *holonomy*. Ehresmann investigates the question of completeness of those locally homogeneous spaces. Under some conditions (in particular, when M is compact an X is a Riemannian homogeneous space), the developing map is a global diffeomorphism and the manifold M is therefore a quotient of X.

While the study of local homogeneous spaces until the 70s was mostly marked by powerful rigidity theorems (Calabi–Weil, Mostow, Margulis...), at the end of the seventies, Thurston resumes Ehresmann's work and proves a very general deformation theorem: if $\mathbf{hol} : \pi_1(M) \to G$ is the holonomy of structure locally modelled on a G-homogeneous space X, then every morphism close enough to $\mathbf{hol}$ is the holonomy of a locally homogeneous structure close to the initial one. This *Ehresmann–Thurston principle*[11] and the use Thurston will make of it in his work on geometrization of 3-manifolds renewed the interest for the geometrization of manifolds. This interest will grow even more during the last decade after the discovery of *Anosov representations*, which are the source of many new examples of locally homogeneous spaces.

Before we precise this notion, I would like to present a few specific geometrization problems that contributed to the development of this research field.

Kleinian groups and hyperbolization of 3-manifolds

Recall tha the Lie group $\mathrm{PSL}(2, \mathbb{C})$ is the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^3$. It acts by homographies on the Riemann sphere, which identifies with the boundary at infinity of $\mathbb{H}^3$.

The *Kleinian groups* are the discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$, and their classification is therefore essentially the classification of complete hyperbolic 3-manifolds. An important subclass, stable under small deformations, is the class of *convex-cocompact Kleinian groups*, which are the fundamental groups of compact hyperbolic 3-manifolds with convex boundary.

11. This terminology is due to Bergeron and Gelander [24], according to whom the theorem can be read between the lines in Ehresmann's work [60].

While the first examples of Kleinian groups appear in the works of Schottky and Klein in relation to the Uniformization of Riemann surfaces, their comprehensive study begins in 1960 with the work of Ahlfors and Bers [3]. In resolving the Beltrami equation under weak regularity hypotheses, they prove that deformations of a convex-cocompact Kleinian group are parameterized by conformal structures on the boundary of the associated hyperbolic manifold.

Some 15 years later, Thurston states his geometrization conjecture for 3-manifolds, which predicts that every closed aspherical and atoroidal 3-manifold has a hyperbolic structure. Building on the work of Ahlfors–Bers, he proves this conjecture for *Haken* manifolds, which can be cut along incompressible surfaces. To simplify a lot, his proof consists in hyperbolizing the manifolds with boundary obtained after cutting along an incompressible surface, then suitably deform the hyperbolic structures on each piece (thanks to the Ahlfors–Bers theorem) in order to glue them together. The proof requires a refined understanding of discrete and faithful surface group representations, for which Thurston develops considerably the Teichmüller theory.

Those results brought Thurston to formulate a number of important conjectures – besides the Hyperbolization conjecture, one can cite the Virtually Haken conjecture and the Ending lamination conjecture – the resolution of which (respectively by Perelman [156], Agol [2], and Brock–Canary–Minsky [33]) lead to a profound understanding of the topology and geometry of 3-manifolds.

Divisible convex sets

The development of hyperbolic and projective geometry during the XIX$^{\text{th}}$ put an end to the supremacy of Euclidean geometry and brought Klein, in his renowned *Erlangen program* [104], to define geometry as the study of homogeneous spaces. When Ehresmann later introduces the notion of locally homogeneous space, he his thus naturally lead to consider more specifically manifolds locally modelled on a homogeneous space.

An interseting subclass of such manifolds is formed by compact quotients of open convex domains in projective spaces. The convex domains with such a cocompact action are called *divisibles.* This notion was introduced by Kuiper on [115], and studied further in Benzécri's thesis [22]. Benzécri thinks that divisble convex sets are rigid objects, and he indeed proves a rigidity theorem under some regularity hypothesis: divisible convex sets with $\mathcal{C}^2$ boundary are projectively equivalent to the projective model of the hyperbolic space. However, a few years later, Katz and Vinberg buid divisible convex sets whose boundary is $\mathcal{C}^1$ but not $\mathcal{C}^2$ [100], while Koszul proves an Ehresmann–Thurston principle for divisble convex sets [114]: if $\Gamma \subset \mathrm{PSL}(n, \mathbb{R})$ divides a convex in $\mathbb{R}\mathbf{P}^n$, then small deformations of Γ keep dividing a convex.[12] One can in particular obtain continuous families of divisible convex sets by deforming a uniform lattice of $\mathrm{SO}(d,1) \simeq \mathrm{Isom}(\mathbb{H}^d)$ inside $\mathrm{SL}(d+1,\mathbb{R})$.

Finally, Choi and Goldman [43] (in dimension 2) and then Benoist [19] (in all dimensions) prove that dividing a convex set is also a closed condition. Therefore, every continuous deformation of a uniform hyperbolic lattice of $\mathrm{SO}(d,1)$ into $\mathrm{SL}(d{+}1,\mathbb{R})$ divides a convex domain. Choi–Goldman's theorem leads in some sense to a classification of divisible convex sets in dimension 2. In contrast, deformation spaces of divisble convex sets in dimension 3 and their relations with the geometrization of 3-manifolds are still obscure, despite the work of Benoist [20] and, more recently, of Ballas–Danciger–Lee [12].

Spacetimes of constant curvature

Another motivation to the study of locally homogeneous spacetimes originates from the theory of relativity. Around 1900, Lorentz, Poincaré and Einstein progressively discover that the symmetry group of the laws of physics is not the group of "gallilean" transformations, but rather the *Lorentz group* $\mathrm{SO}(3,1)$, which preserves Maxwell's equations. A few years later, Einstein incorporates the relations between gravitation and acceleration in the curvature of spacetime: according to the theory of general relativity, our spacetime is a Lorentz manifold whose curvature is related to the *stress-energy tensor* by Einstein's equation.

If one postulates a certain homogeneity to the spacetime[13], it should look like a Lorentzian manifold of constant curvature at large scale. The sign of this curvature, which depends on the value of the cosmological constant and conditions the expansion properties of the universe, has been the topic of more than one controversy in the history of general relativity.

Studying the shape of the universe hence raises a geometrization problem: what are the manifolds locally modelled on a homogeneous Lorentzian manifold of constant curvature? When mathematicians start addressing this question at the end of the 50s, they drift slightly from the physical preoccupations that motivated it. Markus, with Calabi [39] and Auslander [11], initiates in particular the study of closed Lorentzian manifolds of constant curvature. We will come back to the many developments of this topic in Chapter 2.

However, the closedness assumption is not very plausible from a physical point of view because it implies a "time recurrence" that violates the causality

12. Actually, Koszul does not know the Ehresmann–Thurston principle and proved an a priori weaker statement that convex projective structures are open in all projective structures.

13. This is the *perfect cosmological principle*, introduced by Bondi in 1950 and whose relevance can be discussed, which I am not competent for.

principle. It has more sense in physics to think of a *Globally Hyperbolic Cauchy compact* (GHC) spacetime, meaning, very roughly, a spacetime that is compact in space directions bu still satisfies a causality condition.

In his remarkable paper [145], Mess describes the Lorentzian GHC 3-manifolds of constant curvature, and exhibits the deep connections between their deformation spaces and the Teichmüller space of a spacelike surface. Some of these results were later generalized to higher dimensions. In the case of negative curvature, in particular, the GHC manifolds of dimension $d+1$ are (under an additional convexity hypothesis) quotients of an open domain in the *anti-de Sitter space* AdS^{d+1} by the fundamental group of a closed negatively curved manifold of dimension d. Barbot proves in [13] that every continuous deformation of such a group in the isometry group of AdS^{d+1} is again the fundamental group of a GHC anti-de Sitter spacetime.

The paradigm of Anosov groups

The notion of *Anosov subgroup* of a semisimple Lie group G or of *Anosov representation* of a finitely generated group Γ with values in G was introduced by Labourie at the turn of the century [118], and quickly developed by Guichard and Wienhard [84] and many other authors. To paraphrase words that I heard from Anna Wienhard, the theory of Anosov representations provoked a "change of paradigm" in the study of discrete subgroups of Lie groups: while the XX$^{\text{th}}$ century geometry was marked by rigidity theorems (in particular, those of Mostow and Margulis), next to which the few examples of flexible discrete groups (Kleinian groups, symmetries of divisble convex sets...) seemed like disparate counter-examples, the formalism of Anosov groups has transformed this point of view by revealing the richness of flexible discrete groups and by providing a unified framework to their study.

Let us give some precisions. Consider a semisimple Lie group G ($\mathrm{SO}(p,q)$ or $\mathrm{SL}(n,\mathbb{R})$, for instance). The compact G-homogeneous spaces are essentially the *flag varieties*[14] G/P, where P is a *parabolic subgroup*. Let us say that a sequence $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ is P-proximal if it has an attracting fixed point in G/P. Informally, a discrete subgroup Γ of G is P-Anosov if it satisfies some form of uniform P-proximality.

This property (which we state precisely in Chapter 1, Section 1.3.2) has many remarkable geometric and dynamical consequences. It implies that the group Γ is *hyperbolic* in the sense of Gromov [95], and that its boundary at infinity idenfies with a closed Γ-invariant subset of G/P. Guichard and Wienhard also show that a P-Anosov group Γ acts properly discontinuously and cocompactly on and open domain in another flag variety G/Q [84], expanding considerably the list of closed manifolds locally modelled on flag varieties.

14. When $G = \mathrm{SL}(n,\mathbb{R})$, these are indeed the spaces of flags in $\mathbb{R}^n$ of a fixed type.

The main virtue of the Anosov property is its structural stability : small deformations of a P-Anosov group remain P-Anosov. When G is the group of isometries of a hyperbolic space (or more generally when G has rank 1), his unique flag variety is the boundary at infinity of the hyperbolic space, and the Anosov property is then equivalent to convex-cocompactness. The main source of examples of Anosov groups hence consists in deforming convex-cocompact subgroups in rank 1 into higher rank Lie groups. For instance, the deformations of uniform $\mathrm{SO}(d,1)$ lattices inside $\mathrm{SL}(d+1,\mathbb{R})$ or $\mathrm{SO}(d,2)$ (and, more generally, the groups dividing a strictly convex projective domain and the fundamental groups of anti-de Sitter convex-GHC spacetimes) are examples of Anosov groups. Recently, Zimmer [209] and Danciger–Guéritaud–Kassel [52] have proven that the Anosov property always reduced to a *projective convex-cocompactness* property introduced by Crampon–Marquis [50], which unifies convex-cocompact groups in rank 1 and divisble convex sets.

Besides the fact that it provides a unified framework to diverse examples of flexible geometric structures, the main achievement of the theory of Anosov groups is the description of the geometric properties of certain families of surface group representations. Recall that a closed surface Σ of genus at least 2 carries hyperbolic metrics, and that the holonomy of each hyperbolic metric gives a discrete and faithful representation of its fundamental group Γ into $\mathrm{PSL}(2,\mathbb{R})$ called a *Fuchsian representation*. Fuchsian representations modulo conjugation form a connected component of the *character variety* $\mathfrak{X}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ which identifies to the *Teichmüller space* of Σ by the Uniformization theorem. In [118], Labourie showed that *Hitchin representations*, which form a connected component of the character variety $\mathfrak{X}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$, are all Anosov. With Burger, Iozzi and Wienhard, they also proved the same result for *maximal representations* with values in Hermitian Lie groups [35]. These examples gave birth to the *higher Teichmüller theory*, which studies components of Anosov representations inside character varieties of surface groups and their similarities with the Teichmüller space. This field is one of my main research areas.

Ramifications of a research trajectory

My reasearch is initially motivated by geometrization problems, and Anosov representations play a significant role in it. Further on, I chose to organize my work in two themes treated mostly independently: compact quotients of reductive homogeneous spaces on one side and surface group representations on the other.

In this introduction I would like to tell about my work in a chronological rather than logical manner, in order to illustrate how my initial topic of interest (closed locally homogeneous pseudo-Riemannian manifolds) ramified in several autonomous directions. Each paragraph below presents one of these branches.

Pseudo-Riemannian symmetric spaces

I started by thesis by looking at closed manifolds locally modelled on pseudo-Riemannian symmetric spaces. A central question in their study is the question of *completeness*: are those spaces necessarily quotients of their symmetric model? I addressed this question in the case of manifolds locally modelled on a rank 1 Lie group endowed with its Killing metric. This choice was motivated in particular by the fact that compact quotients of those spaces where about to be well understood thanks to the incoming works of Guéritaud–Guichard–Kassel–Wienhard [81, 80], and that it was frustrating not to know whether they were describing in this way all the closed manifolds locally modelled on these Lie groups. I obtained the following result:

Theorem 1 ([186]). *Let G be a Lie group of rank 1 endowed with the action of $G \times G$ by left and right multiplication, and U an open set in G. Assume that there exists a subgroup of $G \times G$ acting properly discontinuously and cocompactly on U. Then $U = G$.*

This very partial result at least implies that compact quotients of G cannot be deformed continuously into incomplete structures.

After a hard time on the problem of completeness, I decided to look at the only rank 1 geometry where it is solved: the case of $G = \mathrm{PSL}(2,\mathbb{R})$, which identifies with the anti-de Sitter space AdS^3. The compact quotients of $\mathrm{PSL}(2,\mathbb{R})$ by a subgroup of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ are described very precisely by the works of Kulkarni–Raymond [117] and Kassel [96]: they are (up to a finite cover) of the form

$$j \times \rho(\Gamma) \backslash \mathrm{PSL}(2,\mathbb{R})$$

where Γ is the fundamental group of a closed surface Σ of genus at least 2, j is a Fuchsian representation and ρ a representation *dominated by* j in the sense that there exists a (j,ρ)-equivariant map $f : \mathbb{H}^2 \to \mathbb{H}^2$ which contracts distances. The remaining problem was to describe the set of such pairs. In particular, is any non-Fuchsian representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ dominated by a Fuchsian representation ?

This question was interesting to Bertrand Deroin because of its relation to another open question: which representations $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ are the holonomy of a *branched hyperbolic structure*? To address these two questions, we first developed an approach based on discrete harmonic maps, which allowed us to see ρ as the holonomy of a "folded" hyperbolic structure with conical singularities of angle $> 2\pi$. This approach (recently completed by Florestan Martin–Baillon [142]) quickly produces a positive answer to the first question if we can handle a certain number of degenerate triangulations. However, after a while, we understood that these difficulties disappear at the cost of a bit of analysis if one replaces discrete harmonic maps by actual harmonic maps. Besides, we gain a better control of the parameters of the construction: each choice of complex structure on Σ produces a Fuchsian representation dominating ρ, and I eventually managed to prove that this produces all dominating representations.

Théorème 10 (Deroin–Tholozan [55, 189])**.** *Let ρ be a non-Fuchsian representation of Γ into* $\mathrm{PSL}(2,\mathbb{R})$ *(actually, into any Lie group of rank* 1*). Then the set of Fuchsian representations dominating ρ modulo conjugation is non-empty and homeomorphic to the Teichmüller space of Σ.*

This theorem provides a description of the *moduli space* of closed anti-de Sitter manifolds of dimension 3.

To conclude the study of those manifolds, I tried to compute their volume. Thanks to the work of Guéritaud–Kassel [81], we know that these manifolds fiber over a closed surface with timelike geodesic fibers. By integrating along the fibers, I obtained the following expression for their volume:

Theorem 2 (Tholozan [191])**.** *Let Γ be a surface group, $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ a Fuchsian representation and $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ a representation dominated by j. Then*

$$\mathbf{Vol}\left(j \times \rho(\Gamma)\backslash \mathrm{PSL}(2,\mathbb{R})\right) = \frac{\pi^2}{2}\left(\mathbf{eu}(j) + \mathbf{eu}(\rho)\right)\ ,$$

where **eu** *denotes the Euler class.*

This result extends readily to quotients of $\mathrm{SO}(d,1)$, using again the fibration given by Guéritaud–Kassel.

The surprising fact that the volume of a closed 3-dimensional manifold was computed by a "characteristic class" required a more conceptual explaination than a down-to-earth computation. I eventually understood how to bypass Guéritaud–Kassel's work by reasoning "up to homotopy", which allowed me to generalize the previous theorem in the following way:

Theorem 3 (Tholozan [191])**.** *Let $\Gamma\backslash G/H$ be the compact quotient of a reductive homogeneous space. Then*

$$\mathbf{Vol}(\Gamma\backslash G/H) = \int_{[\Gamma]} i^*\omega_{G/H}$$

where $\omega_{G/H}$ is a continuous cohomology class of G depending only on H, i is the inclusion of Γ in G and $[\Gamma]$ is a "fundamental class" in the homology of Γ.

I also determined when the class $\omega_{G/H}$ is a characteristic class (implying the rationality of the volume) and deduced from this theorem an explicit volume formula for compact quotients of $\mathrm{SU}(d,1)$. More importantly, I gave several vanishing criteria for $\omega_{G/H}$ which produce powerful obstructions to the existence of compact quotients of certain reductive homogeneous spaces. All this is detailed in Chapter 2.

Surface group representations and harmonic maps

The domination theorem for surface group representations in rank 1 obtained with Bertrand Deroin opened a new research perspective. We naturally wondered what happened of this result for representations in higher rank, starting with Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$.

Let Σ be a closed surface of genus at least 2 and Γ its fundamental group. Each Hitchin representation $\rho : \Gamma \to \mathrm{PSL}(3,\mathbb{R})$ acts properly discontinuously and cocompactly on a convex open domain Ω_ρ in the projective plane and preserves its *Hilbert metric*, whose metric properties capture the algebro-geometric properties of ρ. Results of Crampon [49] and Nie [153] on the *entropy* of those convex domains prove that one cannot hope to dominate uniformly those representations. Getting deeper into the subject, I learnt of the existence of a natural Riemannian metric on each convex domain, the *Blaschke metric*, which comes from the theory of affine spheres. This metric is uniformly comparable to the Hilbert metric and has curvature ≥ -1, suggesting that, contrary to the rank one, Hitchin representations always dominate a Fuchsian one. To refine this result, a precise comparison between Blaschke and Hilbert metrics was missing. I obtained it in [190].

Theorem 4 (Tholozan [190])**.** *Let ρ be a Hitchin representation of a surface group Γ into $\mathrm{PSL}(3,\mathbb{R})$. There exists a Fuchsian representation j such that*

$$L_\rho(\gamma) \geq L_j(\gamma)$$

for all $\gamma \in \Gamma$.

Here, $L_j(\gamma)$ denotes the translation length of $j(\gamma)$ in the hyperbolic plane and $L_\rho(\gamma)$ that of $\rho(\gamma)$ in Ω_ρ with its Hilbert metric. This sharp comparison between surface group representations used once again harmonic analysis. Indeed, the Blaschke metric is related to the unique ρ-equivariant conformal harmonic map from $\widetilde{\Sigma}$ to the symmetric space $\mathrm{PSL}(3,\mathbb{R})/\mathrm{PSO}(3)$.

Encouraged in the idea that twisted harmonic maps can give precise informations on surface group representations, I then moved my interest towards maximal representations into the Hermitian Lie group $\mathrm{SO}(2,d)$. It was becoming impossible to avoid *Higgs bundles*, these holomorphic objects that capture the algebraic properties of twisted harmonic maps into symmetric spaces. Studying carefully the structure of these Higgs bundles, we proved with Brian Collier and Jérémy Toulisse that if $\rho : \Gamma \to \mathrm{SO}(2,d)$ is a maximal representation, any ρ-equivariant conformal harmonic into the symmetric space of $\mathrm{SO}(2,d)$ is the Gauss map of a *maximal spacelike surface* in the pseudo-Riemannian symmetric space $\mathbb{H}^{2,d-1}$. Giving to this maximal surface the role of the affine sphere fo Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$, one obtains many geometric properties of these representations:

Theorem 5 (Collier–Tholozan–Toulisse [45]). *Let $\rho : \Gamma \to \mathrm{SO}(2,d)$ be a maximal representation. Then:*

- *There exists a unique ρ-equivariant conformal harmonic map from $\widetilde{\Sigma}$ to the symmetric space of $\mathrm{SO}(2,d)$, and this map is an embedding.*
- *There exists a Fuchsian representation j such that $L_\rho \geq L_j$.*
- *The domain of discontinuity Ω_ρ of Guichard–Wienhard in the space of isotropic planes of $\mathbb{R}^{2,d}$ admits a ρ-equivariant fibration onto $\widetilde{\Sigma}$ whose fibers are flag submanifolds.*

(Here, $L_\rho(\gamma)$ denotes the logarithm of the spectral radius of $\rho(\gamma)$.)

Those diverse properties and their relevance in higher Teichmüller theory are discussed at length in Chapter 3, Section 3.2.

Highest Teichmüller theory

The main reason for which harmonic maps give such fine descriptions of Hitchin or maximal representations in rank 2 lies in a coincidence: the Higgs bundles associated to conformal harmonic maps are then *cyclic*, a strong structural property that greatly simplifies there study. Starting from rank 3, cyclic Higgs bundles only parametrize a subvariety of the character variety, and it seems difficult to extract from harmonic analysis some precise geometric properties for all representations.

To pursue the study of Anosov representations of surface groups beyond rank 2, it seems more appropriate to approach them with the tools of hyperbolic dynamics. In rough terms, this approach – which can be attributed first to Sambarino [171] – consists in associating to an Anosov representation ρ a *length spectrum* $L_\rho : [\Gamma] \to \mathbb{R}_+$ (where $[\Gamma]$ denotes the conjugacy classes of elements of Γ) which can be seen as the period map of a Hölder reparametrization of the geodesic flow of Σ. The works of Bowen, Margulis or Ruelle on the dynamics of Anosov flows then provide precise informations on the asymptotic behaviour of L_ρ. The *topological entropy* of these flows, which measures the exponential growth rate of L_ρ, plays a crucial role here.

If $\rho : \Gamma \to \mathrm{PSL}(d,\mathbb{R})$ is a Hitchin representation, one can for instance associate to it a flow whose periods are given by

$$L_\rho(\gamma) = \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)) \ ,$$

where λ_i denotes the logarithm of the i^{th} eigenvalue [32]. Bertrand Deroin once pointed out to me that the entropy of such a flow must equal 1 since

L_ρ can be read on the cocycle of derivatives of a $\mathcal{C}^1$ action of Γ on the circle (see also [158]).

While trying to clarify this remark, I understood that there exists a bijective correspondance between the following three spaces:

(1) The space of Hölder reparametrizations of the geodesic flow of Σ with entropy 1, modulo Livšic equivalence,

(2) The space of $\mathcal{C}^{1+Hölder}$ actions of Γ on the circle which are Hölder conjugate to a Fuchsian action, modulo $\mathcal{C}^1$ conjugation,

(3) The space of transversally Hölder foliated hyperbolic metrics on the weakly stable foliation of the geodesic flow of Σ, modulo isotopy.

Besides, this correspondance preserves three natural notions of *period map* from $[\Gamma]$ to $\mathbb{R}_+$.

Je présente cette correspondance dans des notes en cours d'élaboration [192]. Son intérêt est de multiplier les points de vue sur ce que j'aime appeler prétentieusement l'*espace de Teichmüller suprême.* [15] Sa construction n'est que le début d'un programme de recherche qui ambitionne de décrire la géométrie de cet espace de dimension infinie, dans l'espoir d'en déduire des propriétés géométriques des espaces de Teichmüller supérieurs. Le point de vue (3) permet en effet de réintroduire la théorie de Teichmüller classique dans la théorie de Teichmüller supérieure, tandis que le point de vue (2) fournit une analogie entre les représentations Anosov et les représentations maximales dans $\mathrm{Diff}(\mathbb{S}^1)$. Nous donnons plus de détails sur ce projet et ses développements à venir dans la section 3.4.1.

I present this correspondance in some lecture notes [192]. The motivation behing this correspondance is to multiply the points of view on what I like to call pretentiously the *highest Teichmüller space.* [16] Its construction is only the beginning of a program that aims at describing the geometry of this infinite dimensional space, in hope of deducing geometric properties of higher Teichmüller spaces. The point of view (3) allows indeed to reintroduce classical Teichmüller theory in higher Teichmüller theory, while the point of view (2) draws an analogy between Anosov representations and maximal representations into $\mathrm{Diff}(\mathbb{S}^1)$. We give more details on this project and its future developments in Section 3.4.1.

Hilbert Geometries

The study of Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$ allowed me to get a deeper understanding of Hilbert geometries. In particular, my comparison theorem between the Balschke and Hilbert metrics, valid in all dimensions,

15. Pour ma défense, il existe déjà un "espace de Teichmüller universel" et un "super-espace de Teichmüller".

16. For my defense, there already exists a "universal Teichmüller space" and a "super-Teichmüller space".

allowed me to prove a conjecture of Colbois and Verovic on the volume growth of Hilbert geometries:

Theorem 6. *Let Ω be a proper convex domain of the projective space $\mathbb{R}\mathbf{P}^{d-1}$. Then the volume entropy of Ω:*

$$\mathscr{H}(\Omega) \stackrel{\text{def}}{=} \limsup_{R\to+\infty} \frac{1}{R} \left(\log \mathbf{Vol}(B(o,R))\right)$$

is less than $d-2$.

(Here, $B(o,R)$ denotes the ball of radius R about a given point o for the Hilbert metric, and **Vol** denotes its volume for the associated Hausdorff measure.)

When the convex set Ω is divided by a group Γ, the volume entropy coincides with the *critical exponent* of Γ:

$$\delta(\Gamma) = \lim_{R\to+\infty} \frac{1}{R} \log \sharp\{\gamma\in\Gamma \mid d_{Hilb}(o,\gamma\cdot o) \leq R\}\ .$$

In this case, Theorem 6 has been proven first by Crampon with dynamical methods [49]. Crampon proves moreover a rigidity statement: if the entropy equals $d-2$, then the convex domain is and ellipsoid (hence its Hilbert metric is hyperbolic). [17]

These results ask for a more general study of critical exponents of groups Γ that only act convex-cocompactly on a convex domain Ω. The critical exponent of Γ then coincides with the growth of the volume of the convex core and, when Ω is a ball, it equals the Hausdorff dimension of the limit set of Γ in $\partial_\infty\Omega$, by a famous theorem of Sullivan [181]. Daniel Monclair, who was working with Olivier Glorieux on critical exponents for groups acting on the pseudo-Riemannian space $\mathbb{H}^{p,q}$ [70], explained to me that Patterson–Sullivan theory (and its generalization to Gromov hyperbolic spaces by Coornaert [46]) gives in fact an equality between the critical exponent of Γ and the Hausdorff dimension of its limit set for the *Gromov metric* on its boundary. To obtain an inequality with te geometric Hausdorff dimension, it thus suffices to compare the Gromov metric with a Riemannian metric. We eventually obtained:

Théorème 11 (Glorieux–Monclair–Tholozan [71])**.** *Let Γ be a group acting convex-cocompactly on a Gromov hyperbolic proper convex domain $\Omega \subset \mathbf{P}(V)$. Let Λ_Γ be the limit set of Γ in $\partial_\infty\Omega$ and set*

$$\widehat{\Lambda}_\Gamma = \{(x, T_x(\partial_\infty\Omega)), x\in\Lambda_\Gamma\} \subset \mathbf{P}(V)\times\mathbf{P}(V^*)\ .$$

Then the critical exponent of Γ for the Hilbert metric is less than the Hausdorff dimension of $\widehat{\Lambda}_\Gamma$.

This theorem is presented in more details in Section 1.3.3, where we situate in the more general study of critical exponents of Anosov groups.

17. Crampon a priori assumes that Ω is Gromov hyperbolic, but his result has been improved by Barthelmé–Marquis–Zimmer [15] using Theorem 7.

Bounded mapping class group orbits

With Bertrand Deroin, we pursued our investigation of surface groups inside $\mathrm{PSL}(2,\mathbb{R})$ and more specifically the question previously mentioned: Which representations $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ are the holonomies of *branched hyperbolic structures* (i.e. hyperbolic metrics with cone singularities of angles multiple of 2π)? The preliminary results that we obtained required to cut the surface, hyperbolize the pieces and glue back, which lead us to consider the problem of geometrizing surfaces with boundary.

Let $\Sigma_{g,n}$ be a surface of genus g with n boundary components and $\Gamma_{g,n}$ its fundamental group. The *character variety* $\mathfrak{X}(\Gamma_{g,n}, G)$ is essentially the space of of representations of $\Gamma_{g,n}$ into G modulo conjugation. It is foliated by *relative character varieties*, where the conjugacy classes of the images of the boundary curves are fixed, and the *pure mapping class group* $\mathrm{MCG}_{g,n}$ of $\Sigma_{g,n}$ (the group of isotopy classes of homeomorphisms fixing the boundary) acts on $\mathfrak{X}(\Gamma_{g,n}, G)$ and preserves the relative character varieties.

While developing a notion of relative Euler class for representations of $\Gamma_{g,n}$ into $\mathrm{PSL}(2,\mathbb{R})$, we discovered serendipitously the existence of representations of $\Gamma_{0,n}$ whose surprising properties come from the fact that they form compact connected components of certain relative character varieties. In the mean time, Mondello, described the topology of all the relative character varieties into $\mathrm{PSL}(2,\mathbb{R})$ [150]. This encouraged Jérémy Toulisse and I to generalize Mondello's approach via parabolic Higgs bundles to higher rank character varieties. In the end, we obtained the following:

Théorème 12 (Deroin–Tholozan [56], Tholozan–Toulisse [193])**.** *Let G be one of the Lie groups $\mathrm{PU}(p,q)$, $\mathrm{Sp}(2k,\mathbb{R})$ or $\mathrm{SO}^*(2k)$. For all $n \geq 4$, there exists and open set $\Omega \subset \mathfrak{X}(\Gamma_{0,n}, G)$ which is the union of compact connected components of relative character varieties. Moreover, the representations ρ in Ω have the following properties:*

(1) The orbit of $[\rho]$ under the action of $\mathrm{MCG}_{0,n}$ is bounded,

(2) The image by ρ of any simple closed curve on $\Sigma_{0,n}$ has all its eigenvalues of module 1,

(3) For every set D of n points on the Riemann sphere $\mathbb{S}^2$, the ρ-equivariant harmonic map from $\widetilde{\mathbb{S}^2 \backslash D}$ to the symmetric space of G is holomorphic.

This theorem gives in particular many families of bounded mapping class group orbits and allows to formulate conjectures on the general properties of these bounded orbits. This opens a research program, from which we hope to gain in particular some information on finite orbits of $\mathrm{MCG}_{g,n}$, which are related to linear representations of $\mathrm{MCG}_{g,n+1}$. This program and some preliminary results are presented in Section 3.4.3.

Structure of the memoir

The first chapter of this memoir is a very general introduction to the study of discrete subgroups of semisimple Lie groups and their deformations. We start by some background about semisimple Lie groups, their symmetric spaces, their flag varieties and the importance of the Cartan projection in their geometry (Section 1.1). We will then mention some results about linear groups and introduce their *character varieties* (Section 1.2). Finally, we will present the general properties of Anosov groups and we will state in passing our results on Hilbert geometries (Section 1.3).

The rest of my work will be presented in two mostly independent chapters. Chapter 2 surveys our current knowledge about compact quotients of reductive homogeneous spaces. We will mention various obstructions to their existence (Section 2.2) and a few known constructions of such quotients (Section 2.3), then we will discuss a conjecture on their geometry, and explain how it inspired my results on the volume of these spaces. (Section 2.4).

As to Chapter 3, it is devoted to surface group representations. After some precisions about their character varieties where we will in particular introduce higher Teichmüller spaces (Section 3.1), I will introduce harmonic maps, their relation to minimal surfaces and Higgs bundles, and I will explain the role they play in my main results (Section 3.2). In Section 3.3, I will present my constructions of bounded components in relative character varieties. Finally, Section 3.4 will sketch my current research projects regarding surface group representations: "highest Teichmüller theory", branched hyperbolic metrics and bounded mapping class group orbits.

Chapter 1

Preliminaries: discrete subgroups of Lie groups

This chapter tries to present in a synthetic way some classical results about discrete subgroups of Lie groups, which give the context of this memoir. The first section describes briefly the geometry of semisimple Lie groups and their associated symmetric spaces. The second section recalls some important results about finitely generated subgroups of Lie groups and presents the framework to study their deformations. Finally, the third section introduces the Anosov property, which plays a central role in my research.

1.1 Lie groups and symmetric spaces

We begin by introducing the geometry of semisimple Lie groups their associated symmetric spaces and flag varieties. The contents of this section are detailed in many classical books, such as Helgason's *Differential Geometry, Lie Groups and Symmetric Spaces* [87].

1.1.1 Semisimple Lie algebras and Lie groups

Let us start with very classical considerations on Lie groups, mainly to fix some notations and conventions.

Convention 1. All Lie groups and Lie algebras we consider are *real*. Even complex groups such as $\mathrm{SL}(n, \mathbb{C})$ are considered, unless other wise stated, with their underlying real structure.

Let us thus start by recalling that a (real) Lie group G is a smooth manifold with a compatible group structure (i.e. such that multiplication and inverse are smooth maps). We denote by $\mathbf{1}_G$ its identity element. The space of vector fields on G that are invariant under right multiplication forms a finite dimensional Lie subalgebra of the Lie algebra of smooth vector fields, called the *Lie algebra* of G. We will denote it by $\mathrm{Lie}(G)$ or, when it does not bring any confusion, by the same name as the Lie group in gothic character ($\mathfrak{g}$ for Lie(G), $\mathfrak{h}$ for Lie(H)...). The evaluation at $\mathbf{1}_G$ canonically identifies Lie(G) with the tangent space to G at $\mathbf{1}_G$.

Most of the structure of a Lie group is captured by its Lie algebra. To be more precise, let us call two Lie groups *isogenous* if their Lie algebras are isomorphic. Then

- The connected component G_0 of $\mathbf{1}_G$ is a normal Lie subgroup of G isogenous to G,
- When G is connected, the universal cover $\widetilde{G}$ of G has a unique structure of Lie group such that the covering map $\pi : \widetilde{G} \to G$ is a homomorphism. The group $\widetilde{G}$ is isogenous to G, and the kernel of π is a discrete central subgroup of $\widetilde{G}$ isomorphic to the fundamental group of G.

In particular, every Lie group is isogenous to a connected and simply connected Lie group.

Let now $\varphi : G \to H$ be a smooth homomorphism between Lie groups. The differential of φ at $\mathbf{1}_G$ is a Lie algebra homomorphism from Lie(G) to Lie(H). Conversely, every homomorphism from Lie(G) to Lie(H) extends uniquely to a smooth homomorphism from $\widetilde{G}_0$ to H. Of course, the morphism between $\mathrm{d}\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ induced by $\varphi : G \to H$ does not capture anything of the induced morphism of discrete groups $\bar{\varphi} : G/G_0 \to H/H_0$.

When needed, we will assume that our Lie groups are *linear algebraic groups*, i.e. are closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ given by polynomial equations. [1] This implies in particular that G has finitely many components. In any case, we will make the following convention:

Convention 2. All the Lie groups we consider have finitely many connected components.

Remark 1.1.1. It might seem that there is no loss of generality in assuming G connected and simply connected. However, these hypotheses can be incompatible with the algebricity assumption, as shown by the following examples:

- The linear algebraic group $\mathrm{PSL}(2, \mathbb{R})$ has fundamental group $\mathbb{Z}$, and its universal cover is not linear.
- The linear algebraic group $\mathrm{SO}(p, q)$ has 2 connected components, and its neutral component is not an algebraic subgroup.

The action of G on itself by conjugation induces a linear representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ called the *adjoint representation of* G. Differentiating at the identity gives the adjoint representation of $\mathfrak{g}$:

$$\begin{array}{rccl} \mathrm{ad}: & \mathfrak{g} & \to & \mathrm{End}(\mathfrak{g}) \\ & u & \mapsto & \mathrm{ad}_u : v \mapsto [u, v] \ . \end{array}$$

1. An algebraic geometer would more accurately speak of "the real points of a linear algebraic group".

The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form defined by

$$\kappa(u,v) = \mathrm{Tr}(\mathrm{ad}_u \mathrm{ad}_v) \ .$$

The Killing form of $\mathfrak{g}$ vanishes if and only if $\mathfrak{g}$ is *solvable*, i.e. there exists a finite sequence of ideals $\mathfrak{g}_n = \{0\} \subset \mathfrak{g}_{n-1} \subset \ldots \mathfrak{g}_1 \subset \mathfrak{g}_0 = \mathfrak{g}$ such that the successive quotients $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ are abelian. At the opposite, the Killing form is non-degenerate if and only if $\mathfrak{g}$ is *semisimple*, i.e. does not have a non-trivial solvable ideal. In that case, $\mathfrak{g}$ decomposes as a direct sum of Lie algebras that are *simple*, i.e. without non-trivial ideals. This terminology reflects the corresponding terminology for Lie groups. Indeed, the Lie algebra of G is solvable, simple or semisimple if and only if G is isogenous respectively to a solvable group, a simple group or a product of simple groups. In that case, we call G respectively a *solvable Lie group*, a *simple Lie group* or a *semisimple Lie group*.

The Levi decomposition theorem states that every Lie algebra is a semidirect product of a solvable ideal with a semisimple subalgebra called a Levi factor. Consequently, every Lie group is isogenous to the semi-direct product of a normal solvable subgroup with a semisimple subgroup. Once this decomposition is established, Lie group theory branches off in two directions. Here, our focus will be on semisimple Lie groups.

If G is a semisimple Lie group, then the adjoint representation is an isogeny from G to the linear algebraic group $\mathrm{Aut}(\mathfrak{g})$. This isogeny is finite to 1 if and only if G has finite center and finitely many components, which always holds when G is linear algebraic.

Convention 3. Unless otherwise stated, our semisimple Lie groups are assumed to have finite center.

1.1.2 Symmetric spaces of semisimple Lie groups

Let G be a semisimple Lie group (with finite center and finitely many connected components) and $\mathfrak{g}$ its Lie algebra. In order to understand better the structure of G, it is often useful to see G as the isometry group of its *symmetric space*, which we describe here.

A *Cartan involution* of $\mathfrak{g}$ is a Lie algebra automorphism σ such that

— $\sigma^2 = \mathrm{Id}_{\mathfrak{g}}$

— $\kappa(\cdot, \sigma \cdot)$ is negative definite.

The Lie algebra $\mathfrak{g}$ decomposes under σ as

$$\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{g}^{\sigma\perp}$$

where $\mathfrak{g}^\sigma$ is the subalgebra fixed by σ and $\mathfrak{g}^{\sigma\perp}$ is its orthogonal (with respect to the Killing form), in restriction to which $\sigma = -\mathrm{Id}$. We denote by G^σ the subgroup of G whose adjoint action commutes with σ. Because the restriction of $\kappa_{\mathfrak{g}}$ to $\mathfrak{g}^\sigma$ is negative definite, the group G^σ is compact, with Lie algebra $\mathfrak{g}^\sigma$.

As a consequence of the structure theory of semisimple Lie algebras, Cartan showed that every semisimple Lie algebra $\mathfrak{g}$ admits Cartan involutions, which are all conjugate by an element of $\mathrm{Aut}(\mathfrak{g})_0$. (See also [165] and [57] for more direct proofs.) This leads to the following

Definition 1.1.2. The *symmetric space* of G is the space of Cartan involutions of $\mathfrak{g}$.

It follows from Cartan's theorem that the symmetric space of G is homogeneous under the adjoint action of G, and can be identified with the left quotient G/G^σ, where σ is a chosen Cartan involution. The Killing form is positive definite on $\mathfrak{g}^{\sigma\perp}$ and induces a Riemannian metric on G/G^σ which is complete and has non-positive sectional curvature. Moreover, every compact subgroup of G fixes a point in the symmetric space and is therefore conjugate to a subgroup of G^σ, which is thus a *maximal compact subgroup* of G.

The symmetric space G/G^σ is diffeomorphic to a ball. More precisely, the exponential map $\exp : \mathfrak{g} \to G$ induces a diffeomorphism $\exp_\sigma : \mathfrak{g}^{\sigma\perp} \to G/G^\sigma$ which is precisely the exponential map of the geodesic flow of the symmetric Riemannian metric at σ.

1.1.3 Cartan projection

To get a more precise understanding of the relation between the algebraic structure of the group G and the geometry of its symmetric space, one is naturally brought to determine the invariants of pairs of points of G/G^σ under the action of G. This leads to the introduction of the *Cartan projection.*

Cartan subspaces and Weyl chambers

A Cartan subspace of $\mathfrak{g}$ is a maximal abelian subalgebra whose image under the adjoint representation is diagonalizable over $\mathbb{R}$. All the Cartan subspaces are conjugate by an element of $\mathrm{Aut}(\mathfrak{g})$. More precisely, every Cartan subspace is contained in $\mathfrak{g}^{\sigma\perp}$ for some Cartan involution σ, and all the Cartan subspaces contained in $\mathfrak{g}^{\sigma\perp}$ are conjugate by an element of G^σ. The dimension of a Cartan subspace is called the *(real) rank*[2] of G.

Let σ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}^{\sigma\perp}$. The *(restricted) Weyl group*[3] of $\mathfrak{g}$ is the quotient $W = N^\sigma(\mathfrak{a})/Z^\sigma(\mathfrak{a})$ where

$$N^\sigma(\mathfrak{a}) = \{g \in G^\sigma \mid \mathrm{Ad}_g(\mathfrak{a}) = \mathfrak{a}\}$$

2. Unless otherwise precised, "rank" will always refer to the real rank

3. Since all the pairs $(\sigma, \mathfrak{a})$ are conjugate, the abstract structure of the Weyl group is independent of the choice of such a pair.

and

$$Z^\sigma(\mathfrak{a}) = \{g \in N^\sigma(\mathfrak{a}) \mid \mathrm{Ad}_{g|\mathfrak{a}} = \mathrm{Id}\} .$$

The Weyl group is a finite Euclidean reflexion group of $(\mathfrak{a}, \kappa)$ whose fundamental polytope is a *Weyl chamber*. To be more precise, le us recall the definition of the *roots* of G:

Definition 1.1.3. A linear form α on $\mathfrak{a}$ is a *root* if there exists $v \in \mathfrak{g}\backslash\{0\}$ such that

$$[u, v] = \alpha(u)v$$

for all $u \in \mathfrak{a}$.

A vector $u \in \mathfrak{a}$ is called *regular* if $\alpha(u) \neq 0$ for all α in Δ. Each connected component of the set of regular vectors is the interior of a closed convex cone called a *Weyl chamber*. The Weyl group acts simply transitively on the set of Weyl chambers. Moreover, each Weyl chamber is a simplicial cone, and the Weyl group is the Coxeter group generated by the reflections on the walls of a fixed Weyl chamber. This implies in particular that every Weyl group orbit of $\mathfrak{a}$ has a unique representative in a fixed Weyl chamber.

Convention 4. Whenever needed, when given a semisimple Lie group G, we supposed that we have fixed a choice of:
— a Cartan involution σ of $\mathfrak{g}$,
— a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}^{\sigma\perp}$,
— a Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$.

Gathering the above results, one can define the *Cartan projection* in the following way:

Proposition-Definition 1.1.4. *For every $u \in \mathfrak{g}^{\sigma\perp}$, there is a unique vector $\mu(u) \in \mathfrak{a}_+$, called the Cartan projection of u, such that*

$$\mu(u) = \mathrm{Ad}_k(u)$$

for some $k \in G^\sigma$.

For every $g \in G$, there is a unique vector $\mu(g) \in \mathfrak{a}_+$, called the Cartan projection of g, such that

$$g = k_1 \exp(\mu(g)) k_2$$

for some k_1 and $k_2 \in G^\sigma$.

Geometric interpretation

Let us now reinterpretate the notions presented above in terms of the geometry of the symmetric space. We fix a semisimple Lie group G and denote by X its symmetric space. We choose a basepoint $o \in X$ and denote by σ the corresponding Cartan involution [4] of $\mathfrak{g}$, so that $\mathrm{Stab}_G(o) = G^\sigma$.

Note first that the space $\mathfrak{g}^{\sigma\perp}$ identifies canonically with the tangent space to X at o. Expressing the curvature tensor of X in terms of the Lie bracket of $\mathfrak{g}$, one shows that Cartan subspaces of $\mathfrak{g}^{\sigma\perp}$ are exactly the maximal subspaces of T_oX in restriction to which the curvature tensor vanishes. It follows that, for any such Cartan subspace $\mathfrak{a}$, the set $\exp(\mathfrak{a})\cdot o$ is a flat totally geodesic subspace of X. The rank of G is thus the maximal dimension of a flat totally geodesic subspace of its symmetric space.

Now, the Cartan projection $\mu : T_oX \to \mathfrak{a}_+$ classifies the orbits of the action of G^σ on T_oX. Using the transitivity of the action of G, one can extend this Cartan projection to a G-invariant function

$$\mu : TX \to \mathfrak{a}_+$$

with the property that $\mu(u) = \mu(v)$ if and only if there exists $g \in G$ such that $g_*(u) = v$.

Similarly, the Cartan projection $\mu : G \to \mathfrak{a}_+$ factors to a function from $X = G/G^\sigma$ to $\mathfrak{a}_+$ which classifies the orbits of the action of G^σ on X. Using again the transitivity of the action of G one can extend μ to a G-invariant function on $X \times X$ with the property that $\mu(x, y) = \mu(x', y')$ if and only if there exists $g \in G$ such that $g \cdot x = x'$ and $g \cdot y = y'$. The Cartan projection on TX and $X \times X$ correspond to one another via the geodesic flow, in the sense that

$$\mu(u) = \mu(x, \exp_x(u))$$

for all $x \in X$ and all $u \in T_xX$. We hope that giving the same name to these different maps to $\mathfrak{a}_+$ will not bring any confusion.

It is useful to interpretate the Cartan projections on TX and $X \times X$ as "vector valued" metrics on X. Indeed, we have the following proposition:

Proposition 1.1.5. *Let $\|\cdot\|$ be a norm on $\mathfrak{a}$ invariant under the Weyl group. Then the function*

$$\begin{array}{ccc} X \times X & \to & \mathbb{R}_+ \\ (x, y) & \mapsto & \|\mu(x, y)\| \end{array}$$

is a G-invariant distance on X. It is the path distance associated to the G-invariant Finsler metric

$$\begin{array}{ccc} TX & \to & \mathbb{R}_+ \\ u & \mapsto & \|\mu(u)\| \ . \end{array}$$

Conversely, every G-invariant path metric on X is the Finsler metric associated to a W-invariant norm on $\mathfrak{a}$.

4. Strictly speaking, σ and o are the same thing, but it is more convenient to name them differently when seen as an involution of $\mathfrak{g}$ or a point in X.

Opposition involution

The opposite of $\mathfrak{a}_+$ is another Weyl chamber. By transitivity of the Weyl group, there exists an element $w \in W$ such that

$$-w(\mathfrak{a}_+) = \mathfrak{a}_+$$

The involution $\iota = -w$ of $\mathfrak{a}_+$ is called the *opposition involution*.

Depending on G, the opposition involution may or may not be trivial. When it is non-trivial it captures the symmetry defect of the Cartan projection. Indeed, we have:

- For all $g \in G$,
$$\mu(g^{-1}) = \iota(\mu(g))$$
- For all $u \in TX$,
$$\mu(-u) = \iota(\mu(u))$$
- For all $x, y \in X \times X$,
$$\mu(y, x) = \iota(\mu(x, y)) \ .$$

The rank 1 case

When the Lie group G has rank 1, its Cartan subspace is a real line on which the Weyl group acts by $\pm\mathrm{Id}$. Up to scaling, there is thus a unique G-invariant path metric on its symmetric space which is Riemannian. This distance is directly given by the Cartan projection, seen as a function with values in $\mathbb{R}_+$, and the isometry group G is 2-transitive, i.e. for any $x_1, x_2, y_1, y_2 \in X$ such that $d(x_1, x_2) = d(y_1, y_2)$, there exists $g \in G$ mapping x_1 to y_1 and x_2 to y_2.

The Lie group G has rank 1 if and only if its symmetric space has negative sectional curvature. In fact, symmetric spaces of rank 1 are classified, and can all be seen as "hyperbolic spaces over some division algebra". To be more precise, note first that a semisimple Lie group of rank 1 is isogenous to a product of a simple Lie group of rank 1 with a compact group (of rank 0). Let us list the simple Lie groups of rank 1 up to isogeny:

- $\mathrm{SO}(d, 1)$, $d \geq 1$, whose symmetric space is the *hyperbolic space* $\mathbb{H}^d_{\mathbb{R}}$ of real dimension d,
- $\mathrm{SU}(d, 1)$, $d \geq 1$, whose symmetric space is the *complex hyperbolic space* $\mathbb{H}^d_{\mathbb{C}}$, of real dimension $2d$,
- $\mathrm{Sp}(d, 1)$, $d \geq 1$, whose symmetric space is the *quaternionic hyperbolic space* of real dimension $4d$,
- The exceptional Lie group F_4^{-20}, whose symmetric space can be seen as the hyperbolic plane over the octonions.

1.1.4 Roots, parabolic subgroups and flag varieties

We finish this section by recalling some further structure theory for semisimple Lie groups, in order to introduce their *parabolic subgroups* and *flag varieties*. These will play a central role in the study of discrete subgroups of Lie groups, and particularly in the notion of *Anosov subgroup*.

Let us fix a semisimple Lie group G and choose a Cartan involution σ of $\mathfrak{g}$, a Cartan subspace $\mathfrak{a} \subset \mathfrak{g}^{\sigma\perp}$ and a positive Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$. Recall that Δ denotes the set of *roots* of $\mathfrak{g}$. To every root $\alpha \in \Delta$ corresponds an *root space*

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid [u,v] = \alpha(u)v \text{ for all } u \in \mathfrak{a}\} .$$

In particular, the root space $\mathfrak{g}_0$ decomposes orthogonally as $\mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}^\sigma$. The Cartan involution acts as $-\mathrm{Id}$ on $\mathfrak{a}$ and thus maps $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$.

A non-zero root is called *positive* when it takes non-negative values on the Weyl chamber $\mathfrak{a}_+$. A positive root is called *simple* if it cannot be written as the sum of two positive roots. We denote by Δ_+ the set of positive roots and by Δ_S the set of simple roots. Both these sets are preserved by the opposition involution. The set of simple roots is a basis of $\mathfrak{a}^*$, and the positive Weyl chamber is exactly the cone where all the positive roots take non-negative values. Therefore, the Weyl chamber is a simplicial cone, and every simple root vanishes on exactly one of its walls.

Let now Θ be a subset of Δ_S. A vector $u \in \mathfrak{a}_+$ is called Θ-*regular* if $\theta(u) > 0$ for all $\theta \in \Theta$. We call it Θ-*characteristic* if it is Θ-regular and, moreover, $\alpha(u) = 0$ for all $\alpha \in \Delta_S \setminus \Theta$.

Definition 1.1.6.

— The *parabolic subalgebra* associated to Θ is the Lie subalgebra

$$\mathfrak{p}_\Theta = \sum_{\alpha \in \Delta \mid \alpha(u) \geq 0} \mathfrak{g}_\alpha$$

where u is Θ-characteristic.

— The *parabolic subgroup* associated to Θ is the normalizer of $\mathfrak{p}_\Theta$ in G:

$$P_\Theta = \{g \in G \mid \mathrm{Ad}_g(\mathfrak{p}_\Theta) = \mathfrak{p}_\Theta\} .$$

— A subgroup of G is *parabolic* if it is conjugate to P_Θ for some $\Theta \subset \Delta_S$

— The *flag variety* associated to Θ is the compact homogeneous space G/P_Θ, which can be seen as the space of parabolic subgroups of G conjugate to P_Θ.

The parabolic subgroup associated to $\Theta = \emptyset$ is the group G itself. The other parabolic subgroups are called *proper*. If Θ is non-empty, we have

$$P_\Theta = \bigcap_{\theta\in\Theta} P_{\{\theta\}} \ .$$

The parabolic subgroups $P_{\{\theta\}}$ (and their conjugates) are called *maximal parabolic subgroups*. Finally, the parabolic subgroup associated to $\Theta = \Delta_S$ and its conjugates are called *minimal parabolic subgroups*.

One defines similarly the *opposite parabolic subalgebra* $\mathfrak{p}_\Theta^{\mathrm{op}}$ as

$$\mathfrak{p}_\Theta^{\mathrm{op}} = \sum_{\alpha\in\Delta \mid \alpha(u)\leq 0} \mathfrak{g}_\alpha$$

and the *opposite parabolic subgroup* P_Θ^{op} as the centralizer of $\mathfrak{p}_\Theta^{\mathrm{op}}$. It is a parabolic subgroup, conjugate to $P_{\iota(\Theta)}$.

The action of G on $G/P_\Theta \times G/P_{\iota(\Theta)}$ has a unique open orbit: the orbit of $(P_\Theta, P_\Theta^{\mathrm{op}})$. Pairs of parabolic subgroups in this open orbit are called *transverse*.

The example of $\mathrm{SL}(n,\mathbb{R})$

All the above definitions, which are hard to digest for someone not already familiar with Lie theory, are meant to generalize a picture which is quite clear for $\mathrm{SL}(n,\mathbb{R})$.

The Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ is the space $\mathfrak{sl}(n,\mathbb{R})$ of $n\times n$ matrices of trace 0. A canonical choice of Cartan involution for $\mathrm{SL}(n,\mathbb{R})$ is given by $M \mapsto -M^T$, whose stabilizer is the compact subgroup $\mathrm{SO}(n)$ preserving the standard scalar product of $\mathbb{R}^n$. A canonical choice of Cartan subspace $\mathfrak{a}$ is the space of diagonal matrices of trace 0, which has dimension $n-1$. The Weyl group acts on it by permuting the diagonal entries, and a canonical choice of positive Weyl chamber is

$$\mathfrak{a}_+ = \{\mathrm{Diag}(x_1,\ldots,x_n) \mid x_1 \geq \ldots \geq x_n\} \ .$$

For u in $\mathfrak{a}$, define $\varepsilon_i(u)$ as its i^{th} diagonal coefficient. Then:

— The roots of $\mathfrak{sl}(n,\mathbb{R})$ are the linear forms $\varepsilon_i - \varepsilon_j$, $1 \leq i,j \leq n$,

— The positive roots are the linear forms $\varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq n$,

— The simple roots are the linear forms $\theta_i = \varepsilon_i - \varepsilon_{i+1}$,

— The opposition involution maps ε_i to $-\varepsilon_{n-i}$ and thus θ_i to θ_{n-i}.

Let $(e_1,\ldots,e_n)$ denote the canonical basis of $\mathbb{R}^n$. For all $1 \leq k \leq n-1$, define P_k as the stabilizer of $\langle e_1 \ldots, e_k\rangle$ and P_k^{op} as the stabilizer of $\langle e_{k+1},\ldots,e_n\rangle$. Then P_k is the maximal parabolic subgroup $P_{\{\theta_k\}}$ and the associated flag variety $\mathrm{SL}(n,\mathbb{R})/P_k$ is the Grassmannian of k-planes in $\mathbb{R}^n$, that we denote $\mathrm{Grass}_k(\mathbb{R}^n)$. The opposite parabolic subgroup is conjugate to P_{n-k}, and a pair of points in $(\mathrm{SL}(n,\mathbb{R})/P_k) \times (\mathrm{SL}(n,\mathbb{R})/P_{n-k})$ is transverse if and only if the corresponding subspaces of $\mathbb{R}^n$ of dimension k and $n-k$ are in direct sum.

Let now $\Theta = \{\theta_{k_1}, \dots, \theta_{k_l}, k_1 > \dots > k_l\}$ be a subset of Δ_S. Then G/P_Θ is the space of *flags of type* $(k_1, \dots, k_l)$, i.e. increasing chains of subspaces of $F_1 \subset \dots \subset F_l \subset \mathbb{R}^n$ with $\dim(F_i) = k_i$. The opposite flag variety is the space flags of type $(n-k_l, \dots, n-k_1)$. Finally, a flag F of type $(k_1, \dots, k_l)$ and a flag F' of type $(n-k_l, \dots, n-k_1)$ are transverse if and only if

$$F_i \oplus F'_{l-i} = \mathbb{R}^n$$

for all $1 \leq i \leq l$.

Geometric interpretation

When G has rank 1, the set Δ_S contains a single root. There is thus up to conjugation a unique non-trivial parabolic subgroup P. The flag variety G/P is diffeomorphic to a sphere and can be identified with the *boundary at infinity* of the symmetric space X (see Section 1.3.1).

In higher rank, though there is no canonical compactification of X, the flag varieties of G can still be seen as "boundaries" of its symmetric space. For instance:

— The visual boundary of X is a union of closed G orbits parametrized by the projectivization of the positive Weyl chamber. For every $[u] \in \mathbf{P}(\mathfrak{a}_+)$, the corresponding G-orbit in the visual boundary is G/P_Θ, where $\Theta = \{\alpha \in \Delta_S \mid \alpha(u) > 0\}$.

— The symmetric space of $\mathrm{SL}(n,\mathbb{R})$ identifies with the space of positive definite quadratic forms on $\mathbb{R}^n$ modulo scaling, which is a proper convex domain in $\mathbf{P}(\mathrm{Sym}^2(\mathbb{R}^{n*}))$. Its boundary has a unique closed $\mathrm{SL}(n,\mathbb{R})$-orbit consisting of positive quadratic forms of rank 1, which identifies with the flag variety

$$\mathbf{P}(\mathbb{R}^{n*}) \simeq \mathrm{SL}(n,\mathbb{R})/P_{n-1} .$$

— When G is of *Hermitian type*, its symmetric space identifies with a bounded domain in a complex vector space. The boundary of this domain contains a unique closed G-orbit called the *Shilov boundary* of X, which is a flag variety of G.

Dynamical interpretation

Let $u \in \mathfrak{a}_+$ be a Θ-regular vector. Then, almost by construction, P_Θ and P_Θ^{op} are respectively attracting and repelling fixed points for $\exp(u)$ in the corresponding flag varieties.

Elaborating on this remark and using the Cartan decomposition, one can describe the dynamical behaviour of any diverging sequence $(g_n) \in G^{\mathbb{N}}$ in terms of Cartan projections. More precisely, up to extracting a subsequence, assume that

$$\frac{\mu(g_n)}{\|\mu(g_n)\|} \underset{n\to+\infty}{\longrightarrow} v \in \mathfrak{a}_+ \backslash \{0\} .$$

The vector v is Θ-regular for some non-empty subset Θ of Δ_S. Then, up to extracting a further subsequence, we have

Proposition 1.1.7. *There exists a pair $(x_+, x_-) \in G/_{P_\Theta} \times G_{P_{\iota(\Theta)}}$ such that*

$$g_n \cdot x \underset{n\to+\infty}{\longrightarrow} g_+$$

for all $x \in G/P_\Theta$ transverse to x_-.

When G has rank 1, one recovers the *convergence property* of the action of G on the boundary at infinity of its symmetric space.

This gives a further motivation for introducing the flag varieties of G: they are, in some sense, the spaces where the dynamics of discrete subgroups happen.

1.2 Subgroups of Lie groups and their deformations

In a broad sense, this memoir is concerned with the description of (finitely generated) subgroups of Lie groups. Since subgroups of solvable groups are themselves (virtually) solvable, they can mostly be understood with algebraic methods. In contrast, the algebraic properties of subgroups of a semisimple Lie groups (on which we will focus) are more ellusive, and their study belongs to the realms of geometry and dynamics.

1.2.1 Subgroups of semisimple Lie groups

Despite what we just claimed, there are a few general and powerful algebraic properties that are essentially shared by all finitely generated subgroups of a semisimple Lie group. We mention here those that are most relevant, beginning with Malcev's theorem and Selberg's lemma.

Theorem 1.2.1 (Malcev's theorem [134])**.** *Let Γ be a finitely generated subgroup of a linear algebraic group. Then Γ is residually finite, i.e. for all $\gamma \in \Gamma$, there exists a finite index normal subgroup of Γ that does not contain γ.*

Theorem 1.2.2 (Selberg's lemma [174])**.** *Let Γ be a finitely generated subgroup of a linear algebraic group. Then Γ is virtually torsion-free, i.e. there exists a finite index normal subgroup $\Gamma' \subset \Gamma$ such that every $\gamma \in \Gamma' \backslash \{\mathbf{1}_\Gamma\}$ has infinite order.*

When Γ is a discrete subgroup of a semisimple group G, Selberg's lemma has the interesting geometric interpretation that Γ is virtually the fundamental group of a complete non-positively curved manifold. More precisely, Γ acts properly discontinuously on the symmetric space X, and is torsion-free if and only if this action is also free. By Selberg's lemma, one can always find a torsion-free normal subgroup of finite index $\Gamma' \subset \Gamma$. The group Γ' is thus the fundamental group of the complete negatively curved manifold $\Gamma' \backslash X$, and $\Gamma \backslash X$ is the quotient of this manifold by the finite group Γ/Γ'. Finally, Malcev's theorem produces "many" finite coverings of $\Gamma' \backslash X$.

Another spectacular property of finitely generated linear groups is the renowned *Tits alternative*:

Theorem 1.2.3 (Tits)**.** *Let Γ be a finitely generated subgroup of a linear algebraic group. Then, either Γ is virtually solvable, or Γ contains a free group in two generators.* [5]

Zariski closure

The Tits alternative comforts the idea that "generic" subgroups of semisimple Lie groups are necessarily complex from an algebraic, geometric and dynamical point of view. To make sense of this "genericity", let us recall the notion of *Zariski closure.*

Let Γ be a subgroup of a linear algebraic group G. The *Zariski closure* $\bar{\Gamma}^Z$ of Γ is the algebraic subset of G defined by the vanishing of all the polynomial functions that vanish on Γ. It is also the smallest linear algebraic subgroup containing Γ. Note that Γ is virtually solvable if and only if its Zariski closure is a solvable Lie group. We call Γ *Zariski dense* if its Zariski closure contains the identity component of G.

If Γ is not Zariski dense in G, then one typically wants to reduce the study of Γ inside G to that of Γ inside its Zariski closure. Thus, in many situations, there is no loss of generality in considering only Zariski dense subgroups of G. With that in mind, one can for instance reformulate the Tits alternative in the following way:

Theorem 1.2.4 (Tits)**.** *Let G be a semisimple linear algebraic group and Γ a finitely generated Zariski dense subgroup. Then Γ contains a free group in 2 generators.*

Remark 1.2.5. It might sometimes be convenient to talk about Zariski closures in a semisimple Lie group G which is not assumed algebraic. We define it by

$$\bar{\Gamma}^Z \overset{\text{def}}{=} \mathrm{Ad}^{-1}\left(\overline{\mathrm{Ad}(\Gamma)}^Z\right) ,$$

5. Note that the free group in two generators contains a free group in k generators for all k.

where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation.

1.2.2 Deformations and character varieties

In this memoir, we will be particularly interested in "deforming" discrete subgroups Γ of a Lie group G. To make this notion more precise, one should consider Γ as an abstract group that we can *represent* as a subgroup of G in possibly different ways. The inclusion $i : \Gamma \hookrightarrow G$ is an example of such a representation, which we will try to deform continuously. A first way to construct such deformations is to conjugate the representation by a continuous familly of transformations in G. Such deformations, however, are somehow irrelevant both from a geometric and dynamical point of view. We will thus call them "trivial deformations". These considerations will lead us to introduce the space of representations of an abstract (finitely generated) group Γ into G and its quotient under conjugation.

Berforehand, let us settle on some more heuristic terminology which will be used throughout the memoir. Let Γ be a finitely generated group and G a Lie group.

- A *small deformation* of a representation $\rho : \Gamma \to G$ is a representation $\rho' : \Gamma \to G$ that belongs to a small neighbourhood of ρ in $\mathrm{Hom}(\Gamma, G)$ (endowed with the topology of pointwise convergence).
- A *continuous deformation* of a representation $\rho : \Gamma \to G$ is (depending on the context) a representation $\rho' \in \mathrm{Hom}(\Gamma, G)$ belonging to the connected component of ρ, or a continuous path in $\mathrm{Hom}(\Gamma, G)$ starting at ρ. It will then be called *smooth* if the image of each $\gamma \in \Gamma$ varies smoothly along the path.
- If Γ is a subgroup of G, a *small/continuous deformation of* Γ into G is a small/continuous deformation of the inclusion $i : \Gamma \hookrightarrow G$.
- A small/continuous deformation ρ' of ρ is called *trivial* if there exists $g \in G$ such that $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ for all $\gamma \in \Gamma$.
- A representation $\rho : \Gamma \to G$ is *locally rigid* if every small deformation of ρ is trivial. A subgroup $\Gamma \subset G$ is locally rigid if the inclusion $i : \Gamma \hookrightarrow G$ is locally rigid.

Character varieties

Let Γ be a finitely generated group (e.g. the fundamental group of a compact manifold) and G a semisimple Lie group. We denote by $\mathrm{Hom}(\Gamma, G)$ the set of homomorphisms – or *representations* – from Γ to G, endowed with the topology of pointwise convergence. Given a finitely generating set S, the embedding

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, G) & \to & G^S \\ \rho & \mapsto & (\rho(s))_{s\in S} \end{array}$$

identifies $\mathrm{Hom}(\Gamma, G)$ with a real anaytic subset which is a real algebraic variety when G is linear algebraic. In particular, $\mathrm{Hom}(\Gamma, G)$ is locally connected by smooth arcs.

The group G acts on $\mathrm{Hom}(\Gamma, G)$ by conjugating representations:

$$g \cdot \rho : \gamma \to g\rho(\gamma)g^{-1} .$$

We denote the quotient of $\mathrm{Hom}(\Gamma, G)$ under this action by $\widehat{\mathfrak{X}}(\Gamma, G)$ and call it the *naïve character variety* of Γ into G. It is naïve because the conjugation action of G is typically not proper, so that the quotient $\widehat{\mathfrak{X}}(\Gamma, G)$ is not a Hausdorff topological space. We denote by $\mathfrak{X}(\Gamma, G)$ the *largest Hausdorff quotient* of $\widehat{\mathfrak{X}}(\Gamma, G)$ and call it the *character variety* of Γ into G.

Though the action of G on $\mathrm{Hom}(\Gamma, G)$ is not proper, it is not too wild either (because it is essentially algebraic), and one can understand rather explicity which conjugacy orbits are identified in the character variety. To state this precisely, let us introduce first some terminology. A Lie subgroup H of G is called *reductive* if its Lie algebra $\mathfrak{h}$ is non-degenerate with respect to the Killing form of $\mathfrak{g}$. The *Zariski closure* of a representation ρ is the Zariski closure of its image.

Definition 1.2.6. A representation $\rho : \Gamma \to G$ is called *reductive* if its Zariski closure is reductive. It is *irreducible* if it is reductive and its centralizer in G is compact.

These definitions generalize classical definitions for linear representations. Indeed, when $G = \mathrm{SL}(V)$, irreducible representations are the ones that do not preserve a non-trivial proper subspace of V, and reductive representations are those that decompose into direct sums of irreducible representations.

We have the following theorem:

Theorem 1.2.7.
- — *The closure of every conjugacy orbit contains a unique closed conjugacy orbit.*
- — *The conjugacy orbit of $\rho \in \mathrm{Hom}(\Gamma, G)$ is closed if and only if ρ is reductive.*
- — *Two representations ρ and ρ' are identified in $\mathfrak{X}(\Gamma, G)$ if and only the closures of their conjugacy orbits intersect.*

Corollary 1.2.8. *The character variety $\mathfrak{X}(\Gamma, G)$ identifies with the set of conjugacy classes of reductive representations.*

The reductive representation contained in the closure of the orbit of ρ is obtained in the following way. Let P be the smallest parabolic subgroup of G containing the image of ρ, let L be a Levi factor of P and $p : P \to L$ the projection morphism. Then $p \circ \rho$ is reductive and is a limit of conjugates of ρ. In particular, ρ is reductive if and only if it takes values in a Levi factor of P, and ρ is irreducible if and only if its image is not contained in a proper parabolic subgroup of G. Hence Zariski dense representations are irreducible.

Interpretation via Geometric Invariant Theory

In this specific paragraph, G is a complex semisimple linear algebraic group. In that case, the above construction is just the topological counterpart of the algebraic construction of the quotient $\mathrm{Hom}(\Gamma, G)//G$ via Geometric Invariant Theory.

The space $\mathrm{Hom}(\Gamma, G)$ is now a complex affine variety (which can in fact be defined over $\mathbb{Q}$). Let $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ denote its algebra of regular functions and $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ the subalgebra of G-invariant regular functions. Then $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is the algebra of regular functions of a complex affine variety called the *GIT quotient* of $\mathrm{Hom}(\Gamma, G)$ and denoted $\mathrm{Hom}(\Gamma, G) /\!\!/ G$. It turns out that $\mathrm{Hom}(\Gamma, G) /\!\!/ G$ *is* the character variety. More precisely, there is a homeomorphism $\Phi : \mathfrak{X}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G) /\!\!/ G$ (for the analytic topology) such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, G) & & \\ \downarrow & \searrow & \\ \mathfrak{X}(\Gamma, G) & \xrightarrow{\Phi} & \mathrm{Hom}(\Gamma, G) /\!\!/ G \end{array} .$$

Unfortunately, this nice algebraic description does not generalize well when the Lie group is real, for the following reason. Let $G_{\mathbb{R}}$ be a real algebraic group and $G_{\mathbb{C}}$ the group of its complex points. Then the inclusion $G_{\mathbb{R}} \to G_{\mathbb{C}}$ induces a map from $\mathfrak{X}(\Gamma, G_{\mathbb{R}})$ to the real points $\mathfrak{X}(\Gamma, G_{\mathbb{C}})^{\mathbb{R}}$ of the complex character variety. Morally, if $\mathfrak{X}(\Gamma, G_{\mathbb{R}})$ were the nice real algebraic variety that we hoped, then this map would be an isomorphism. However:

— It is not surjective in general, because $\mathfrak{X}(\Gamma, G_{\mathbb{C}})^{\mathbb{R}}$ contains the image of $\mathfrak{X}(\Gamma, G'_{\mathbb{R}})$ for other real forms $G'_{\mathbb{R}}$ of $G_{\mathbb{C}}$,

— It is not injective in general, because there might be some outer automorphism τ of $G_{\mathbb{R}}$ which is the restriction of an inner automorphism of $G_{\mathbb{C}}$, so that $[\rho]$ and $[\tau \circ \rho]$ have the same image in $\mathfrak{X}(\Gamma, G_{\mathbb{C}})$.

Example 1.2.9. Let Γ be the fundamental group of a closed surface of genus $g \geq 2$. Then the real points of the character variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$ consist of conjugacy classes of representations with values in $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{PSU}(2)$. The character variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}(2, \mathbb{R}))$ has $4g-3$ connected components that are classified by the *Euler class* (see Section 3.1.4). However, the conjugation by the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$ preserves $\mathrm{PSL}(2, \mathbb{R})$ and reverses the Euler class so that the components of Euler class $\pm k$ are identified in $\mathfrak{X}(\Gamma, \mathrm{PSL}(2, \mathbb{C}))$.

While this example illustrates some subtleties that ought to be taken seriously, it should not prevent us from thinking of $\mathfrak{X}(\Gamma, G_\mathbb{R})$ has a nice space. Richardson and Slodowy [166], extending the work of Kempf and Ness [102] to real groups, proved for instance that $\mathfrak{X}(\Gamma, G_\mathbb{R})$ is a real semi-algebraic set and that the map $\mathfrak{X}(\Gamma, G_\mathbb{R}) \to \mathfrak{X}(\Gamma, G_\mathbb{C})$ is finite to one and proper. Acosta described more precisely the real points of $\mathfrak{X}(\Gamma, G_\mathbb{C})$ for classical Lie groups [1].

Tangent space

Informally, the tangent space to $\mathfrak{X}(\Gamma, G)$ at a representation ρ is the vector space generated by first derivatives of small deformations of ρ modulo first derivatives of trivial deformations. To make this more precise, let $(\rho_t)_{t\in[0,\varepsilon)}$ be a smooth deformation of ρ and define

$$\begin{array}{rccl} u: & \Gamma & \to & \mathfrak{g} \\ & \gamma & \mapsto & \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left(\rho_t(\gamma)\rho(\gamma)^{-1}\right) \end{array} .$$

Deriving the relation $\rho_t(\gamma\eta) = \rho_t(\gamma)\rho_t(\eta)$ gives the following cocycle relation for u:

$$u(\gamma\eta) = u(\gamma) + \mathrm{Ad}_{\rho(\gamma)} u(\eta) \ .$$

Assume now that $\rho_t = \mathrm{Ad}_{g(t)} \circ \rho_0$ for some smooth curve (g_t) in G such that $g_0 = \mathrm{Id}_G$. Setting $v = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g_t$, one finds

$$u(\gamma) = \mathrm{Ad}_\rho(v) - v \ .$$

Formally, the tangent space to $\mathfrak{X}(\Gamma, G)$ at ρ is thus the *first cohomology group with twisted coefficients* $H^1(\Gamma, \mathrm{Ad}_\rho)$. This is in fact more than a formal computation: at a smooth point of $\mathfrak{X}(\Gamma, G)$, the topological tangent space is indeed $H^1(\Gamma, \mathrm{Ad}_\rho)$.

Note that the twisted cohomology group $H^1(\Gamma, \mathrm{Ad}_\rho)$ can be interpreted as a character variety into an affine group. Indeed, $u : \Gamma \to \mathfrak{g}$ is a twisted cocycle for Ad_ρ if and only if the map

$$\begin{array}{ccl} \Gamma & \to & \mathrm{Aff}(\mathfrak{g}) \\ \gamma & \mapsto & [x \mapsto \mathrm{Ad}_\rho(\gamma)x + u(\gamma)] \end{array}$$

is a homomorphism. Moreover, two cocycles differ by a coboundary if and only if the corresponding affine actions on $\mathfrak{g}$ are conjugate by a translation. This correspondence between affine actions and infinitesimal deformations of linear representations is at the heart of Danciger–Guéritaud–Kassel's construction of proper affine actions of non-solvable groups [51, 53].

1.2.3 Topological invariants

In this section we assume for convenience that Γ is the fundamental group of a manifold M. To every representation $\rho : \Gamma \to G$, one can associate a principal G-bundle

$$M \times_\rho G \stackrel{\text{def}}{=} (\widetilde{M} \times G)/\Gamma$$

where Γ acts on $\widetilde{M} \times G$ by

$$\gamma \cdot (x, g) = (\gamma \cdot x, \rho(\gamma) g) \ .$$

This principal bundle is endowed with a flat connection, induced by the trivial connection on $\widetilde{M} \times G$.

Conversely, every flat principal G-bundle over M is isomorphic to $M \times_\rho G$, where $\rho : \Gamma \to G$ is its *holonomy.* However, this holonomy is only defined up to conjugation, and provides a bijection between isomorphism classes of principal G-bundles over M and the naïve character variety $\widehat{\mathfrak{X}}(\Gamma, G)$. This bijection is sometimes referred to as the *Riemann–Hilbert correspondence.*

Now, one can forget the flat connection to obtain a map from the $\mathfrak{X}(\Gamma, G)$ to the set of isomorphism classes of principal G-bundles. The following classical proposition states that this map is locally constant.

Proposition 1.2.10. *If ρ' is a continuous deformation of ρ, then $M \times_\rho G$ and $M \times_{\rho'} G$ are isomorphic as principal G-bundles.*

This remark opens the possibility to use topological invariants of principal G-bundles to discriminate between connected components of $\mathfrak{X}(\Gamma, G)$.

Reduction of structure group

Let K be a maximal compact subgroup of G, so that $X = G/K$ is the symmetric space of G. One can associate to $\rho : \Gamma \to G$ the X-bundle $M \times_\rho X = (M \times_\rho G)/K$. A function $f : \widetilde{M} \to X$ factors to a section of $M \times_\rho X$ if and only if it is ρ-*equivariant*, i.e.

$$f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$$

for all $\gamma \in \Gamma$ and all $x \in \widetilde{M}$.

Now, since X is contractible, such a section always exists and is moreover unique up to homotopy. It gives a *reduction of structure group* from $M \times_\rho G$ to a principal K-bundle, which is well-defined up to isomorphism. The topological classification of G-bundles hence reduces to the classification of principal K-bundles.

Obstruction theory

A first family of invariants comes from trying to construct recursively a section of the principal bundle $M \times_\rho G$ over the k-skeleton of a cellular decomposition of M to the $k+1$-skeleton. The first obstruction controls the possiblity of finding a section over the 1-skeleton. It is simply the representation $\bar{\rho} : \Gamma \to G/G_0$, which is trivial if and only if ρ takes values in G_0.

Assume now that the first obstruction is trivial. We can thus find a section over the 1-skeleton. A second obstruction appears when trying to extend this section over the 2-skeleton. This obstruction lives in $\mathrm{H}^2(M, \pi_1(G_0))$ and vanishes if an only if the representation ρ can be lifted to a representation into $\widetilde{G}_0$.

Example 1.2.11. The Lie group $\mathrm{PSL}(2,\mathbb{R})$ is connected and has fundamental group $\mathbb{Z}$. Therefore, if M is a closed oriented surface, the only obstruction to the triviality of a principal $\mathrm{PSL}(2,\mathbb{R})$-bundle over M lives in $\mathrm{H}^2(M,\mathbb{Z}) = \mathbb{Z}$, and is called the *Euler class* of the principal bundle. The Euler class of a representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is, by definition, the Euler class of the associated principal $\mathrm{PSL}(2,\mathbb{R})$-bundle.

Example 1.2.12. For $n \geq 3$, we have

$$\pi_1(\mathrm{SL}(n,\mathbb{R})) = \pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z} .$$

One can thus associate to a principal $\mathrm{SL}(n,\mathbb{R})$-bundle over M and to a connected component of $\mathrm{Hom}(\Gamma, \mathrm{SL}(n,\mathbb{R}))$ a cohomology class in $\mathrm{H}^2(M,\mathbb{Z}/2\mathbb{Z})$, called its *second Stiefel–Whitney class*.

Characteristic classes

Another source of topological invariants is given by the theory of characteristic classes.

Let H be a connected semisimple Lie group. Let EH be a contractible space on which H acts freely and properly. Then $BH = EH/H$ is a classifying space for H, and EH is the universal principal H-bundle over BH. Let now E be a principal H-bundle over M. Then there exists a continuous map $f : M \to BH$ – called a classifying map – such that E is isomorphic to f^*EH and, moreover, this map is unique up to homotopy. The *characteristic classes* of E are the pull-back of the cohomology classes of BH to M.

Certain obstruction classes can be recovered in this way. For instance, when $H = \mathrm{SO}(2)$, there is a class in $\mathrm{H}^2(BH,\mathbb{Z})$ whose pull-back by f gives the Euler class of E. Similarly, when $H = \mathrm{SO}(n)$, the Stiefel–Whitney class of E is the pull-back by f of a class in $\mathrm{H}^2(BH,\mathbb{Z}/2\mathbb{Z})$.

The *Chern–Weil homomorphism* provides a differential geometric way to compute the characteristic classes of E with real coefficients. Recall that, if

∇ is a principal connection on E, then its curvature F_∇ is a 2-form on M with values in the $\mathfrak{h}$-bundle associated to E. Now, given P a Ad_H-invariant homogeneous polynomial of degree l on $\mathfrak{h}$, one can evaluate P on F_∇ to obtain a closed $2l$-form on M. Moreover, the de Rahm cohomology class of this form does not depend on the choice of the connection ∇.

Let $\mathbb{R}[\mathfrak{h}]^H$ denote the algebra of Ad_H-invariant polynomials on $\mathfrak{h}$.

Theorem 1.2.13 (Chern–Weil homomorphism)**.** *There exists a homomorphism*

$$CW_H : \mathbb{R}[\mathfrak{h}]^H \to \mathrm{H}^\bullet(BH, \mathbb{R})$$

such that, for any principal H-bundle E over a manifold M with classifying map $f : M \to BH$, for any principal connection ∇ on E, and for any homogeneous polynomial $P \in \mathbb{R}[\mathfrak{h}]^H$,

$$[P(F_\nabla)] = f^* CW_H(P) \ .$$

Moreover, if H is compact, then CW_H is an isomorphism.

Remark 1.2.14. Since every principal H-bundle E admits a reduction of structure group to a maximal compact subgroup K, the characteristic classes of E are those of its compact reduction, which are all obtained via Chern–Weil theory of compact groups. It is sometimes useful, however, to consider the Chern–Weil homomorphism of a non-compact group.

Assume now that K is a maximal compact subgroup of a semisimple Lie group G. Then G is a principal K-bundle over $X = G/K$ which carries a G-invariant connection ∇. Every homogeneous polynomial $P \in \mathbb{R}[\mathfrak{k}]^K$ thus defines a G-invariant closed form $P(F_\nabla)$ on X that we call a *Chern–Weil form.* Let $\Omega^\bullet(X)^G$ denote the algebra of G-invariant forms[6] on X.

Finally, Let ρ be a representation of Γ into G. The G-invariant forms on X can be pulled-back to M by any section of the X-bundle $M \times_\rho X$, and the cohomology class of this pull-back is independent of the chosen section. We denote abusively this operation by ρ^*. Let now E be the reduction of $M \times_\rho G$ to a principal K-bundle, and let $f : M \to BK$ be a classifying map for E.

Proposition 1.2.15. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{R}[\mathfrak{k}]^K & \longrightarrow & \Omega^\bullet(G/K)^G \\ {\scriptstyle CW_K}\downarrow & & \downarrow{\scriptstyle \rho^*} \\ H^\bullet(BK, \mathbb{R}) & \xrightarrow{f^*} & H^\bullet(M, \mathbb{R}) \ . \end{array}$$

6. By a theorem of Cartan, every invariant form on a symmetric space is closed.

In other words, the characteristic classes of (the K-reduction of) $M \times_\rho G$ can be computed by pulling back G-invariant forms on the symmetric space.

In general, there are G-invariant forms on G/K that are not Chern–Weil forms. These can still be pulled back under ρ to get cohomology classes on M. Though such classes have been studied in particular cases (see for instance [34]), it does not seem to have been established in full generality that they are locally constant on the character variety. We found a general proof of this fact which consists in interpreting these other classes as *Chern–Simons characteristic classes* associated to pairs of connections. This will be discussed again in Section 2.4.2.

1.2.4 Dynamical invariants

While topological invariants identify connected components of character varieties, some refined invariants of dynamical nature can capture more information about how the properties of a discrete group vary within a continuous family. We introduce them here in a very general context, but their relevance will appear later on when we focus on Anosov groups.

Let us first recall some very general definitions: Let (X, d_X) be a geodesic metric space and g an isometry of X. The *translation length* of g is defined by

$$l(g) \overset{\text{def}}{=} \inf_{x \in X} d(x, g \cdot x) \ .$$

When the distance d_X has some good convexity properties (e.g. when X is CAT(0)), the translation length of an isometry coincides with its *stable length*:

$$l(g) = \lim_{n \to +\infty} \frac{1}{n} d(o, g^n \cdot o) \ ,$$

where o is any point in X. Both the translation length and the stable length are invariant under conjugation.

Now, let Γ be a finitely generated group. We denote by $[\Gamma]$ the set of conjugacy classes in $\Gamma \backslash \{\mathbf{1}_\Gamma\}$.

Definition 1.2.16. The *length spectrum* of a representation $\rho : \Gamma \to \mathrm{Isom}(X)$ is the function

$$\begin{array}{rccc} L_\rho : & [\Gamma] & \to & \mathbb{R}_+ \\ & [\gamma] & \mapsto & l(\rho(\gamma)) \ . \end{array}$$

When the representation ρ is discrete with finite kernel, we define its *entropy* as the exponential growth rate of its length spectrum, namely:

Definition 1.2.17. The *entropy* of a representation $\rho : \Gamma \to \mathrm{Isom}(X)$ is the quantity

$$\mathscr{H}(\rho) \overset{\text{def}}{=} \limsup_{R \to +\infty} \frac{1}{R} \log \sharp\{\gamma \in [\Gamma] \mid L_\rho^X(\gamma) \leq R\} \ .$$

The entropy often coincides with the *critical exponent* of ρ, defined has the exponential growth rate of orbits:

$$\delta(\rho) \stackrel{\text{def}}{=} \limsup_{R\to+\infty} \frac{1}{R} \log \sharp\{\gamma \in \Gamma \mid d(o,\gamma,o) \leq R\} \ .$$

Though these quantities a priori take value in $[0,+\infty]$, we will soon focus on situations where they are finite and non-zero.

The notions of length spectrum and entropy are motivated by the following situation: assume Γ is the fundamental group of a closed Riemannian manifold (M, g_M) of negative curvature, and ρ is given by the action of Γ on the universal cover $\widetilde{M}$. Then the set $[\Gamma]$ is in bijection with closed orbits of the geodesic flow of M, and the length spectrum L_ρ is simply given by the length of these closed orbits. Finally, the entropy $\mathscr{H}(\rho)$ equals the topological entropy of this geodesic flow, and also coincides with the exponential growth rate of the volume of balls in the universal cover:

$$\delta(\rho) = \mathscr{H}(\rho) = \lim_{R\to+\infty} \frac{1}{R} \log \mathbf{Vol}(B_{\widetilde{M}}(o,R)) \ .$$

Let us now consider G a semisimple Lie group, with Cartan projection $\mu : G \to \mathfrak{a}_+$. In order to associate a length spectrum to representations into G, it is natural to consider G as acting on its symmetric space X. Recall, however, that X admits many G-invariant Finsler metrics which all derive from the Cartan projection. This motivates the introduction of a "vector" length spectrum. Let us first introduce the *Jordan projection*:

Definition 1.2.18. The *Jordan projection* on G is the function

$$\begin{array}{rccl} \lambda : & G & \to & \mathfrak{a}_+ \\ & g & \mapsto & \lim_{n\to+\infty} \frac{1}{n}\mu(g^n) \end{array}$$

where μ is the Cartan projection of G.

Example 1.2.19. When $G = \mathrm{SL}(n,\mathbb{C})$, the Jordan projection associates to a matrix g the logarithms of the modules of its eigenvalues in decreasing order.

The Jordan projection is a conjugacy invariant. Let now ρ be a representation of a finitely generated group Γ into G. We define its *vector valued length spectrum* as the function

$$\begin{array}{rccl} \vec{L}_\rho : & [\Gamma] & \to & \mathfrak{a}_+ \\ & \gamma & \mapsto & \lambda(\rho(\gamma)) \ . \end{array}$$

On can compose $\vec{L}_\rho$ with any adequate function N on $\mathfrak{a}_+$ to obtain some notion of length spectrum for ρ. In particular, if N is a W-invariant norm on $\mathfrak{a}_+$, then the length spectrum

$$L^N(\rho) : \gamma \mapsto N(\vec{L}_\rho(\gamma))$$

is the length spectrum of ρ acting on the symmetric space equiped with the distance associated to N. Each choice of N gives a corresponding notion of entropy $\mathscr{H}^N$. Quint studied in a very general context the dependence of $\mathscr{H}^N$ on N when N is a positive linear form on $\mathfrak{a}_+$ [161].

1.2.5 Rigidity

Before investigating deformations of discrete groups any further, let us pause for a moment and ask which groups admit such deformations. This brings the question of local rigidity of representations, which has been widely investigated in the case of lattices. We briefly recall here those rigidity results.

Recall that a representation $\rho : \Gamma \to G$ is *locally rigid* if every small deformation of ρ is conjugate to ρ. When ρ is irreducible, this is equivalent to ρ being an isolated point in $\mathfrak{X}(\Gamma, G)$. The representation ρ is *infinitesimally rigid* if $H^1(\Gamma, \mathrm{Ad}_\rho) = \{0\}$, which implies local rigidity.

Recall that a *lattice* in a Lie group G is a discrete subgroup Γ such that $\Gamma\backslash G$ has finite volume. A lattice is called *uniform* if $\Gamma\backslash G$ is compact. Selberg, Calabi, Weil, Garland–Ragunathan and Margulis successively established the infinitesimal rigidity of lattices in most simple Lie groups.

Theorem 1.2.20 (Weil[202], Garland–Ragunathan [65], Margulis [138])**.** *Let Γ be a lattice in a simple Lie group G. If G is not isogenous to $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{PSL}(2, \mathbb{C})$, then the inclusion $\Gamma \hookrightarrow G$ is infinitesimally (hence locally) rigid.*

In contrast, torsion-free lattices in $\mathrm{PSL}(2, \mathbb{R})$, which are fundamental groups of hyperbolic surfaces of finite volume, are easily deformed by deforming the hyperbolic structure of the surface. The case of $\mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{Isom}(\mathbb{H}^3)$ is more subtle. While uniform lattices in $\mathrm{PSL}(2, \mathbb{C})$ are locally rigid, non-uniform torsion-free lattices have a complex k-dimensional family of deformations, where k is the number of cusps of the corresponding quotient of $\mathbb{H}^3$. However, these deformations are never discrete and faithful. Indeed, Mostow proved that any discrete and faithful representation $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ of a lattice Γ is conjugate to the inclusion [152].

In higher rank, the infinitesimal rigidity of lattices follows from the stronger *super-rigidity theorem* of Margulis, of which we cite the local version:

Theorem 1.2.21 (Margulis's local super-rigidity [138])**.** *Let Γ be a lattice in a simple Lie group G of rank at least 2. Then every representation ρ of Γ into another Lie group H is infinitesimally rigid.*

This theorem was extended to lattices in $\mathrm{Sp}(n, 1)$ and F_4^{-20} by Corlette [48]. It implies in particular that such lattices cannot even be deformed inside a larger group.

In contrast, there exist lattices in $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ that can be deformed non-trivially into a larger group. We will come back to those examples in Section 2.3, where they form the main source of deformations of compact Clifford–Klein forms. Finally, lattices in $\mathrm{PSL}(2,\mathbb{R})$ have many deformations in all semisimple Lie groups. The vast and well-developed theory of their character varieties will occupy the third chapter of this memoir.

1.3 Discreteness and the Anosov property

In a strict sense, a "deformation of a discrete group" $\Gamma \subset G$ should be understood as a small deformation of the inclusion that remains discrete and faithful. In general, however, discreteness is not an open property, This motivates the search for sufficient (and, more difficult, necessary) conditions which would guarantee the stability of the discreteness property.

1.3.1 Quasi-isometric embeddings in rank 1

A map f between two metric spaces (X, d_X) and (Y, d_Y) is *quasi-isometric* if there exists a constant $C > 1$ such that, for all $x, y \in X$,

$$\frac{1}{C} d_X(x,y) - C \leq d_Y(f(x), f(y)) \leq C d_X(x,y) + C \ .$$

We will call it *a quasi-isometry* if, moreover, every $y \in Y$ is at uniformly bounded distance from a point in the image of f.

Let Γ be a group generated by a finite set S symmetric under $s \mapsto s^{-1}$. The length $|\gamma|_S$ of an element γ is the length of the shortest word with letters in S representing Γ. One can then define a left invariant distance on Γ by setting $d_S(\gamma, \eta) = |\gamma^{-1}\eta|$. This distance is induced by the path metric of the *Cayley graph* associated to S, whose vertices are the elements of Γ and where two vertices γ and η are connected by an edge of length 1 if and only if $\gamma^{-1}\eta \in S$. The Cayley graphs associated to different choices of finite generating sets are quasi-isometric.

Let now G be a semisimple Lie group and o an arbitrary basepoint in its symmetric space X.

Definition 1.3.1. A representation $\rho : \Gamma \to G$ is a *quasi-isometric embedding* if there exists a constant $C > 1$ such that

$$d(o, \gamma \cdot o) \geq \frac{1}{C} |\gamma|_S - C$$

for all $\gamma \in \Gamma$. A subgroup Γ of G is quasi-isometrically embedded if the inclusion $\Gamma \hookrightarrow G$ is a quasi-isometric embedding.

One easily verifies that:
— the definition does not depend on the basepoint o,

— it is equivalent to $\rho : (\Gamma, d_S) \to G$ being quasi-isometric (for some left invariant metric on G).

The quasi-isometric property is particularly powerful when the group G has rank 1, which can be explained by the fact that the symmetric space X is then *Gromov hyperbolic*, a notion which behaves well with respect to quasi-isometries. [7]

Let us first recall very briefly the definition of Gromov hyperbolicity:

Definition 1.3.2. A geodesic metric space (X, d) is *hyperbolic* (in the sense of Gromov) if there exists $\delta \geq 0$ such that for any geodesic triangle T in X, there exists a point at distance less than δ from all the sides of T.

A locally compact group Γ is *hyperbolic* (in the sense of Gromov) if it admits a proper and cocompact action on a hyperbolic geodesic metric space.

Example 1.3.3. A Riemannian comparison theorem of Alexandrov, together with an elementary property of hyperbolic geometry, implies that the symmetric spaces of rank 1 are Gromov hyperbolic.

A Gromov hyperbolic space X can be compactified by its *boundary at infinity* $\partial_\infty X$, which can be defined as the set of equivalence classes of geodesic rays in X, where two rays are equivalent if they remain at bounded distance. When X is the symmetric space of a rank one Lie group G, its boundary at infinity is the unique non-trivial flag variety G/P.

The strength of Gromov's notion of hyperbolicity comes from its invariance under *quasi-isometries*. As a consequence, a finitely generated group is hyperbolic if and only if its Cayley graph (for any finite generating set) is hyperbolic. Moreover, quasi-isometric maps between Gromov hyperbolic spaces extend to continuous injective maps between their boundaries at infinity. In particular, one can define the boundary at infinity $\partial_\infty \Gamma$ of a hyperbolic group Γ as the boundary of any of its Cayley graphs. These properties have strong consequences on quasi-isometric embeddings into rank 1 Lie groups.

Theorem 1.3.4 (See for instance [28])**.** *Let Γ be a finitely generated group, G a semisimple Lie group of rank 1 and $\rho : \Gamma \to G$ a quasi-isometric embedding. Then:*

— *[Hyperbolicity] The group Γ is Gromov hyperbolic.*

— *[Boundary map] The map $\gamma \mapsto \rho(\gamma) \cdot o$ extends continuously to an injective ρ-equivariant boundary map*

$$\xi_\rho : \partial_\infty \Gamma \to \partial_\infty X \ .$$

7. This presentation is anachronic: Gromov's notion of hyperbolicity is in fact a way to abstract properties of negatively curved spaces and their isometries in a purely group-theoretic context. Nonetheless, hyperbolicity in the sense of Gromov also provides a very practical way to understand coarse properties of negatively curved spaces.

— *[Domain of discontinuity]* ρ *acts properly discontinuously and cocompactly on* $\Omega_\rho = \partial_\infty X \setminus \xi_\rho(\partial_\infty \Gamma)$.

— *[Convex-cocompactness]* ρ *acts properly discontinuously and cocompactly on a non-empty closed convex subset of* X.

Moreover, the quasi-isometric property is stable under deformations of the representation.

Theorem 1.3.5. *Let* G *be a semisimple Lie group of rank* 1 *and* $\rho_0 : \Gamma \to G$ *a quasi-isometric embedding. Then there exists a neighbourhood* U *of* ρ_0 *in* $\mathrm{Hom}(\Gamma, G)$ *such that every* ρ *in* U *is a quasi-isometric embedding. Moreover, for every* $x \in \partial_\infty\Gamma$*, the map*

$$\begin{array}{rcl} U & \to & \partial_\infty X \\ \rho & \mapsto & \xi_\rho(x) \end{array}$$

is analytic.

Quasi-isometrically embedded subgroups of rank one Lie groups appeared first in the context of *Kleinian groups*, i.e. discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ (and, by extension, of any semisimple Lie group of rank 1). In the language of Kleinian groups, quasi-isometrically embedded groups are called *convex-cocompact*. When Γ is convex-cocompact, the image of its boundary map coincides with its *limit set* Λ_Γ, whose complement is the *maximal domain of discontinuity* Ω_Γ.

Let us finally mention that, for Kleinian groups in $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{Isom}_+(\mathbb{H}^3)$, Theorem 1.3.5 admits a converse:

Theorem 1.3.6 (Sullivan [182])**.** *Let* Γ *be a finitely generated group and* ρ *a discrete and faithful representation of* Γ *into* G *such that every small deformation of* ρ *remains discrete and faithful. Then, either* ρ *is rigid or* ρ *is a quasi-isometric embedding.*

Entropy and Patterson–Sullivan theory

For convex cocompact groups in rank 1, the length spectrum and entropy have a dynamical interpretation.

Let G be a semisimple Lie group of rank 1 with symmetric space X, and Γ a torsion-free convex-cocompact subgroup of G. In this setting, Jordan projections take values into $\mathbb{R}_+$, so that there is (up to scaling) a unique notion of length spectrum L_ρ for the inclusion $\rho : \Gamma \hookrightarrow G$. Denote by $T_1(\Gamma\backslash X)$ the unit tangent bundle to $\Gamma\backslash X$, endowed with the geodesic flow φ. The set $[\Gamma]$ is in bijection with the set of closed orbits of φ (with multiplicity), and the length spectrum simply associates its length to each closed orbit.

The closure of the union of all closed orbits is the non-wandering set $RT_1(\Gamma\backslash X)$ of the geodesic flow. It is compact and its lift to $T_1 X$ is the union of all geodesics with endpoints in the limit set Λ_Γ. The works of Bowen and Margulis on hyperbolic dynamics [29, 138] give the following counting estimate on the growth of the length spectrum:

Theorem 1.3.7 (see for instance [171]). *We have the following*

$$\sharp\{\gamma \in [\Gamma] \mid L_\rho(\gamma) \leq R\} \sim_{R\to+\infty} \frac{e^{\mathscr{H}(\varphi)R}}{R} ,$$

where $\mathscr{H}(\varphi)$ is the topological entropy of the geodesic flow φ on $RT_1(\Gamma\backslash X)$. In particular,

$$\mathscr{H}(\rho) = \mathscr{H}(\varphi) .$$

Patterson [155] and Sullivan [181] developed the theory of conformal measures on the limit set Λ_Γ, which provides a more geometric understanding of the Bowen–Margulis measure of the flow φ. A striking consequence of their work is the following:

Theorem 1.3.8 (Sullivan [181]). *Let Γ be a convex-cocompact subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$. *Then the entropy of Γ equals the Hausdorff dimension of Λ_Γ.*

Here, the metric on $\mathbb{H}^n$ is normalized to have curvature -1. For other rank one Lie groups, the entropy still coincides with the Hausdorff dimension of the limit set endowed with the *Gromov metric* on the boundary [46], which does not coincide with the visual metric in variable curvature.

1.3.2 Anosov properties in higher rank

Quasi-isometric embeddedness is not as robust a notion in higher rank (see [84] for counter-examples). On the other hand, it was discovered during the last decades of the XX$^{\text{th}}$ century that, in various geometric situations, some of the good properties of convex-compact groups in rank 1 remained true when the group is deformed into a higher rank Lie group (see for instance [83]). At the turn of the century, Labourie introduced the notion of *Anosov representation* [118] (for a uniform lattice in a rank one Lie group), which turned out to give a unifying point of view on those disparate situations. His original definition was designed to deduce the stability of this property from the topological stability of hyperbolic dynamical systems. Since then, the notion has been extended to all Gromov hyperbolic groups [84], and an alternative definition has been given which is more synthetic and more geometric. Moreover, Kapovich–Leeb–Porti proved that this definition does not a priori require the group to be hyperbolic [95]. The many recent works on Anosov representations have contributed to comfort the idea that they are indeed the right generalization of quasi-isometric embeddedness in higher rank Lie groups.

Recall that a particular feature of rank 1 Lie groups is that they have a unique flag variety G/P on which their action has fairly simple dynamics:

every diverging sequence in G has a unique attracting point. In contrast, in higher rank, the dynamics of a diverging sequence on the various flag varieties depend on the asymptotic direction of the Cartan projections $\mu(g_n)$. This explains why the Anosov property is a notion relative to a choice of parabolic subgroup, and might give an intuition of why it has to involve a precise control on the Cartan projections.

Let G be a semisimple Lie group, Θ a non-empty subset of the set of simple roots of $\mathfrak{g}$. Let also Γ be a group generated by a finite symmetric set S.

Definition 1.3.9. A representation $\rho : \Gamma \to G$ is called Θ-*Anosov* or P_Θ-*Anosov* if there exists a constant $C > 1$ such that, for all $\gamma \in \Gamma$ and all $\theta \in \Theta$,

$$\theta(\mu(\gamma)) \geq \frac{1}{C}|\gamma|_S - C \ .$$

A subgroup Γ of G is called Θ-Anosov if the inclusion $\Gamma \hookrightarrow G$ is Θ-Anosov.

In less precise words, the Anosov property not only assumes that the Cartan projection of $\gamma \in \Gamma$ grows linearly with γ (which would be equivalent to quasi-isometric embeddedness); it also requires this Cartan projection to become linearly far from the walls of the Weyl chamber defined by the simple roots in Θ.

Using the simple fact that Γ is invariant under taking inverses, one easily shows that ρ is Θ-Anosov if and only if it is Θ^{op}-Anosov. The following theorem, which combines results of [118, 84, 95], asserts that Anosov representations have properties very similar to quasi-isometric embeddings in rank 1:

Theorem 1.3.10. *Let $\rho : \Gamma \to G$ be a Θ-Anosov representation. Then:*

— *[hyperbolicity] The group Γ is Gromov hyperbolic.*

— *[Boundary maps] There exist two continuous ρ-equivariant maps ξ_ρ and ξ_ρ^{op} from $\partial_\infty\Gamma$ to G/P_Θ and G/P_Θ^{op} respectively, which are injective and transverse, meaning that $\xi_\rho(x)$ and $\xi_\rho^{\mathrm{op}}(y)$ are transverse for all $x \neq y$.*

— *[Domains of discontinuity] For certain parabolic subgroups P, there exists an open ρ-invariant domain $\Omega_\rho \subset G/P$ on which ρ acts properly discontinuously and cocompactly.*

Remark 1.3.11. An important difference with the rank one case is that the flag varieties containing a domain of discontinuity are not necessarily the ones in which the boundary curves live. For example, if $\rho : \Gamma \to \mathrm{SL}(4, \mathbb{R})$ is P_2-Anosov, then its boundary map ξ_ρ takes values into the Grassmanian of 2-planes, but it admits a domain of discontinuity in $\mathbb{R}\mathbf{P}^3$, consisting of the complement of all the projective lines $[\xi_\rho(x)]$.

The strength of the Anosov property and source of many examples is its stability under small deformations:

Theorem 1.3.12. *Let $\rho_0 : \Gamma \to G$ be a Θ-Anosov representation. Then there exists a neighbourhood U of ρ_0 in $\mathrm{Hom}(\Gamma, G)$ such that every ρ in U is Θ-Anosov. Moreover, for every $x \in \partial_\infty \Gamma$, the map*

$$\begin{array}{rcl} U & \to & (G/P_\Theta) \times (G/P_{\Theta^{\mathrm{op}}}) \\ \rho & \mapsto & (\xi_\rho(x), \xi_\rho^{\mathrm{op}}(x)) \end{array}$$

is analytic.

In particular, if Γ is quasi-isometrically embedded in a rank one subgroup H of a higher rank group G, then Γ is Θ-Anosov for some choice of Θ (because the Cartan projections of Γ belong to a one dimensional ray of a higher dimensional Weyl chamber), and remains so after a small (possibly Zariski dense) deformation into the ambient group G. This observation has shed a new light on some geometric constructions such as Koszul's deformations of divisible convex sets [114].

1.3.3 Convex-cocompactness in Hilbert geometries

The convex-cocompactness property does not extend to higher rank in a straightforward way. Indeed, Quint [162] and Kleiner–Leeb [105] proved that the only discrete subgroups of a higher rank simple Lie group that act cocompactly on a closed convex subspace of its symmetric space are the uniform lattices. Nonetheless, it was discovered recently that Anosov groups satisfy a form of *projective convex-cocompactness* introduced initially by Crampon and Marquis [50].

A *proper convex domain* Ω of $\mathbb{R}\mathbf{P}^{n-1}$ is an open domain which is convex and bounded in some affine chart. Proper convex domains carry a complete projectively invariant Finsler metric called the *Hilbert metric*, for which projective segments are geodesic. When Ω is the interior of a ball, the Hilbert metric is Riemannian of constant sectional curvature -1 and one recovers the Klein model of the hyperbolic space. More generally, a proper convex domain is called *hyperbolic* if its Hilbert metric is hyperbolic in the sense of Gromov. (Hyperbolicity criteria for proper convex domains have been given by Benoist in [18].)

Definition 1.3.13. A group $\Gamma \subset \mathrm{PSL}(n, \mathbb{R})$ is *projectively convex-cocompact* if there exists a Γ-invariant hyperbolic proper convex domain Ω of $\mathbb{R}\mathbf{P}^{n-1}$ and a non-empty, Γ-invariant, closed convex subset C of Ω on which Γ acts properly discontinuously and cocompactly. [8]

8. In [52], these groups would be called *strongly* projectively convex-cocompact, to make the distinction with a weaker notion that does not assume Ω to be hyperbolic. We won't discuss this weaker notion here.

The following theorem was proven independently by Danciger–Guéritaud–Kassel [52] and Zimmer [209]:

Theorem 1.3.14. *Let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a projectively convex-cocompact group. Then Γ is P_1-Anosov. Conversely, if Γ is P_1-Anosov and preserves a proper convex domain, then Γ is projectively convex-cocompact.*

The assumption that Γ preserves a proper convex domain is not as strong as it seems. First, it is invariant under continuous deformations withing the set of P_1-Anosov representions. Moreover, let i denote the representation of $\mathrm{SL}(n, \mathbb{R})$ given by its action on quadratic forms of $\mathbb{R}^n$. Then for every P_1-Anosov subgroup Γ of $\mathrm{SL}(n, \mathbb{R})$, the group $i(\Gamma)$ is still P_1-Anosov and preserves the proper convex domain of positive definite quadratic forms. It is thus projectively convex-cocompact. Finally, every Θ-Anosov subgroup of G can be made P_1-Anosov by composing with a well-chosen linear representation of G (depending only on Θ). Therefore, every Anosov group or representation can be seen as projectively convex-cocompact after composing with a well-chosen linear representation.

Length spectra and entropies

Recall that Θ-Anosov representations into higher rank Lie groups have a vector valued length spectrum, from which one can derive various notions of length spectrum by evaluating some nice functions on the Weyl chamber. This includes evaluating simple roots $\theta \in \Theta$, which are not positive on $\mathfrak{a}_+$ but still grow linearly with the word length according to the Anosov property.

To be more concrete, let us specialize to P_1-Anosov representations. Let thus Γ be a finitely generated group and $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ a P_1-Anosov representation. We define the *highest weight length spectrum* of ρ by

$$L_\rho^{hw} : \gamma \mapsto \frac{\lambda_1(\rho(\gamma)) - \lambda_d(\rho(\gamma))}{2}$$

and the *simple weight length spectrum* by

$$L_\rho^{sw} : \gamma \mapsto \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)) \; .$$

(Here, $\lambda_i = \varepsilon_i \circ \lambda$ denotes the i^{th} component of the Jordan projection.) The highest weight length spectrum is the length spectrum corresponding to some natural invariant Finsler metric on the symmetric space, while the simple weight length spectrum quantifies in some sense the P_1-Anosov property. We also denote by $\mathscr{H}^{hw}(\rho)$ and $\mathscr{H}^{sw}(\rho)$ the corresponding entropies, which are non-zero and finite.

Assume furthermore that ρ acts convex-cocompactly on a Gromov hyperbolic convex domain $\Omega_\rho \subset \mathbb{R}\mathbf{P}^{n-1}$. Then L_ρ^{hw} coincides with the length spectrum of ρ seen as a representation to $\mathrm{Isom}(\Omega_\rho, d_{Hilb})$, and $\mathscr{H}^{hw}(\rho)$ equals the topological entropy of the geodesic flow of $\rho(\Gamma)\backslash\Omega_\rho$ restricted to its non-wandering set.

A number of works have been devoted to understanding the relation between critical exponents and Hausdorff dimension of limit sets in this setting. In particular, Crampon proved in [49] that, when $\rho(\Gamma)$ acts cocompactly on Ω_ρ, the highest weight entropy of ρ is less than $d-2$, with equality if and only if ρ is conjugate to a representation into $\mathrm{SO}(d-1,1)$. In [71], Glorieux, Monclair and I obtained the following generalization, which slightly extends the previous work [70] of the first two authors :

Theorem 1.3.15 (Glorieux–Monclair–Tholozan [71])**.** *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a projectively convex-cocompact representation. Then*

$$\mathscr{H}^{hw}(\rho) \leq HDim((\xi_\rho, \xi_\rho^*)(\partial_\infty\Gamma)) \leq \mathscr{H}^{sw}(\rho),$$

where $(\xi_\rho, \xi_\rho^*)(\partial_\infty\Gamma)$ denotes the image of $\partial_\infty\Gamma$ into $\mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d^*})$ under the boundary maps ξ_ρ and ξ_ρ^*.

The right inequality was obtained independently by Pozzetti–Sambarino–Wienhard in [160], where they also give conditions for it to be an equality. In their subsequent work [159], they strengthen those results and show in particular that, when $\rho(\Gamma)$ acts cocompactly on Ω_ρ, then

$$\mathscr{H}^{sw}(\rho) = n-2\ .$$

We conjecture a rigidity result analogous to Crampon's when the left inequality holds, but this result is out of reach so far. A more accessible statement would be the following

Conjecture 1.3.16. *If $\mathscr{H}^{hw}(\rho) = \mathscr{H}^{sw}(\rho)$, then ρ is conjugate to a representation into $\mathrm{SO}(d-1,1)$.*

Finally, let us mentioned that Crampon's theorem is equivalent to an inequality on the volume growth of Hilbert geometries of divisible convex sets, which Colbois and Verovic conjectured should hold without any group action [44]. In [190], I proved this conjecture using a comparison lemma between the Blaschke and Hilbert metrics (Lemma 3.2.26).

Theorem 1.3.17 (Tholozan [190])**.** *Let Ω be a proper convex set of $\mathbb{R}\mathbf{P}^{d-1}$. Then there exists a constant $C > 0$ such that, for all $x \in \Omega$ and all $R \geq 1$,*

$$\mathbf{Vol}(B(x,R)) \leq C\ e^{(d-2)R}\ ,$$

where $B(x,R)$ denotes any ball of radius R for the Hilbert metric, and **Vol** *computes its volume with respect to the associated Hausdorff measure (or any other coarsely equivalent volume form).*

This result was then recovered by Vernicos and Walsh with less analytic methods [201].

Chapter 2

Compact quotients of reductive homogeneous spaces

In this chapter, we investigate compact quotients of reductive homogeneous spaces. After reviewing the various known obstructions to their existence, we will explain how to construct some examples and describe in more details the few examples that admit deformations. These will turn out to be deeply connected to Anosov groups.

2.1 Compact quotients of homogeneous spaces

Let X be a smooth connected manifold endowed with a faithful and transitive action of a Lie group G. One can identify X with the right quotient G/H, where H is the subgroup of G fixing a basepoint in X. A *compact quotient* of X is a quotient of X under the left action of a subgroup Γ of G acting properly discontinuously and cocompactly on X. By Selberg's lemma, up to taking a finite index subgroup, one can always assume that Γ acts freely on X, so that $\Gamma\backslash X$ is a closed manifold.

Our investigation of compact quotients of homogeneous spaces is guided by the following general questions:

Question 2.1.1. *Does a given homogeneous space X admit compact quotients ?*

Question 2.1.2. *What is the topology and geometry of the compact quotients of X ?*

Question 2.1.3. *Do these quotients admit deformations, and can we describe their deformation spaces ?*

We will make these questions more precise throughout this section, starting with the case best understood of a Riemannian homogeneous space.

2.1.1 Riemannian homogeneous spaces and standard quotients

We call the homogeneous space X *Riemannian* if G preserves a Riemannian metric on X. This happens if and only if the isotropy subgroup H is compact. When X is Riemannian, the discrete surbgroups of G acting properly discontinuously and cocompactly on X are exactly the uniform lattices of G, which have been extensively studied throughout the XX$^{\text{th}}$ century. In particular, Borel and Harish-Chandra constructed uniform arithmetic lattices in every semisimple Lie group G [27]. This answers Question 2.1.1 for Riemannian symmetric spaces.

Concerning Question 2.1.2, one can remark that $\Gamma\backslash G/H$ fibers over $\Gamma\backslash G/K$, where K is a maximal compact subgroup of G containing H. The quotient $\Gamma\backslash G/K$ is a closed aspherical manifold by the Cartan–Iwasawa–Malcev theorem. A lot more can be said about the topology and geometry of Riemannian locally symmetric spaces (see for instance [23]), which have been and still are an active research topic. From our perspective, however, we consider those as "well-understood".

Finally, by Calabi–Weil's local rigidity theorem (cf Theorem 1.2.20), almost all uniform lattices in semisimple Lie groups are locally rigid. The exception to this rule is formed by the uniform lattices Γ in $\mathrm{PSL}(2,\mathbb{R})$, the deformation spaces of which identify with the *Teichmüller spaces* of surfaces (see Section 3.1.1). This essentially answers Question 2.1.3 in the Riemannian setting.

Even for a non-Riemannian homogeneous space X, the existence of compact quotients sometimes reduces to an existence theorem for lattices. Indeed, if some Lie subgroup L of G acts properly and cocompactly on X, then any uniform lattice in L will then act properly discontinuously and cocompactly on X. This leads to the following definition:

Definition 2.1.4. A compact quotient $\Gamma\backslash G/H$ is called *standard* if Γ is virtually a uniform lattice in a connected Lie subgroup L of G (which must then act properly and cocompactly on G/H).

When G is solvable, every discrete subgroup of G is virtually a uniform lattice in some connected Lie subgroup, hence every compact quotient of G/H is standard. We will therefore focus on the opposite case where G is semisimple and H is a reductive subgroup. The homogeneous space $X = G/H$ is then called *reductive*.

Remark 2.1.5. By focusing on reductive homogeneous spaces, we are deliberately putting aside some geometries where G is neither solvable nor reductive, notably the affine geometry (where $G = \mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ and $H = \mathrm{GL}(n,\mathbb{R})$). Since the incorrect proof of Auslander [10], it is conjectured that all compact quotients of the affine space are standard (and more precisely, are virtually solvmanifolds). Margulis [137], and more recently Danciger–Guéritaud–Kassel [53] have constructed many non-standard proper actions on the affine space (which are not cocompact). They moreover uncovered some subtle relations between quotients of the affine space and quotients of certain reductive homogeneous spaces. Unfortunately, we will not discuss this topic further.

2.1.2 Pseudo-Riemannian geometry of reductive homogeneous spaces

From now on, G denotes a semisimple Lie group with finitely many connected components, and H a reductive Lie subgroup of G which does not contain a non-trivial normal subgroup, so that G acts faithfully on the reductive homogeneous space $X = G/H$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote respectively the Lie algebras of G and H. Recall that H is a reductive subgroup when $\mathfrak{h}$ is non-degenerate with respect to the Killing form of G. The quotient space $\mathfrak{g}/\mathfrak{h}$ thus identifies with $\mathfrak{h}^\perp$, and the restricted Killing form on $\mathfrak{h}^\perp$ induces a G-invariant pseudo-Riemannian metric on G/H. We denote by $(\dim_+(G/H), \dim_-(G/H))$ its signature.

There always exists a Cartan involution σ of G preserving H. We fix it once and for all and denote by G^σ and H^σ its fixed points in G and H respectively. Then $\dim_-(G/H) = \dim(G^\sigma) - \dim(H^\sigma)$ and the quotient space G^σ/H^σ is a closed totally geodesic negative definite subspace of maximal dimension. moreover, all such subspaces are translates of G^σ/H^σ by some element of G, and G/H deformation retracts to G^σ/H^σ.

First main example: pseudo-Riemannian hyperbolic spaces

A first enlightening example is the pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q}$. Let $\mathbb{R}^{p,q+1}$ denote the vector space $\mathbb{R}^{p+q+1}$ endowed with the standard quadratic form of signature $(p, q+1)$:

$$\mathbf{q}(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q+1}^2 \ .$$

Then the space $\mathbb{H}^{p,q} \subset \mathbf{P}(\mathbb{R}^{p,q+1})$ is defined as the set of $\mathbf{q}$-negative lines.

It is homogeneous under the action of the group $\mathrm{PO}(p, q+1)$ of linear automorphisms of $\mathbf{q}$ modulo $\pm\mathrm{Id}$, and this action preserves a pseudo-Riemannian metric of signature (p, q) and constant negative sectionnal curvature (see for instance [204]). The stabilizer of the point $[0, \ldots, 0, 1]$ is isomorphic to the subgroup $\mathrm{O}(p, q)$. The space $\mathbb{H}^{p,q}$ thus identifies with the reductive homogeneous space

$$G/H = \mathrm{PO}(p, q+1)/\mathrm{O}(p, q) \ .$$

In particular, the space $\mathbb{H}^{p,0}$ is the usual (Riemannian) hyperbolic space $\mathbb{H}^p_{\mathbb{R}}$, and the Lorentzian space $\mathbb{H}^{p,1}$ is the *anti-de Sitter space*, that we will also denote AdS^{p+1}.

Let V be a negative definite subspace of dimension $q+1$ in $\mathbb{R}^{p,q+1}$. Then $\mathbf{P}(V) \subset \mathbb{H}^{p,q}$ is a compact timelike totally geodesic subspace of $\mathbb{H}^{p,q}$ of dimension q. These subspaces are the translates of G^σ/H^σ introduced above. Transversally, if W is a subspace of $\mathbb{R}^{p,q+1}$ of signature $(p,1)$, then $\mathbf{P}(W)$ intersects $\mathbb{H}^{p,q}$ in a totally geodesic spacelike submanifold of dimension p isometric to the hyperbolic space $\mathbb{H}^p_{\mathbb{R}}$. This "compactness in the timelike directions" and "contractibility in the spacelike directions" gives a good first intuition of the pseudo-Riemannian geometry of reductive homogeneous spaces.

Second main example: group spaces

Let H be a semisimple Lie group. The Killing form on $\mathfrak{h}$ can be extended to a bi-invariant pseudo-Riemannian metric κ_H on H. Its isometry group is isogenous to $H \times H$ acting on H by

$$(h_1, h_2) \cdot x = h_1 x h_2^{-1}$$

and the stabilizer of $1_H \in H$ is the diagonal subgroup

$$\Delta(H) = \{(h,h) \mid h \in H\} .$$

The space (H, κ_H) is thus the reductive homogeneous space $H \times H/\Delta(H)$. We call such a space a *group space*.

Let σ be a Cartan involution of H. Then the subgroup of fixed points H^σ is a compact totally geodesic timelike subspace of H of maximal dimension, and all the other such spaces have the form $h_1 H^\sigma h_2^{-1}$ for some $h_1, h_2 \in H$.

2.1.3 Locally homogeneous manifolds and compact quotients

Quotients of X are a priori a particular case of manifolds *locally modelled on* X mentioned in the introduction of this memoir, which we call from now on (G,X)*-manifolds*, following Thurstion. A (G,X)-manifold is a manifold endowed with an atlas of charts with values in X and whose coordinate changes are elements of G. If G is the group of diffeomorphisms of X preserving some geometric structure (a metric, a connection), then one should think of (G,X)-manifolds as manifolds carrying a geometric structure locally isomorphic to that of X. For instance, manifolds locally modelled on $\mathbb{H}^{p,q}$ are those carrying a pseudo-Riemannian metric of signature (p,q) and constant negative curvature.

A (G,X)-structure on a manifold M can alternatively be described as a pair $(\mathbf{dev}, \mathbf{hol})$, where $\mathbf{hol}$ is a homomorphism from $\pi_1(M)$ to G called the *holonomy representation* and $\mathbf{dev}$ is a $\mathbf{hol}$-equivariant local diffeomorphism from $\widetilde{M}$ to X called the *developing map*. In particular, a quotient of X is a (G, X)-manifold. Such (G, X)-manifolds are called *complete*. This terminology, which dates back to Ehresmann [59], comes from its close relation to geodesic completeness. I discussed this topic extensively in my thesis [187].

The most striking result of the theory of (G, X)-structures is the *Ehresmann–Thurston principle*, which states that holonomy representations are stable under small deformations:

Theorem 2.1.6 (Ehresmann–Thurston principle [194, 24])**.** *Let X be a G-homogeneous space and M be a closed manifold of the same dimension as X endowed with a (G, X)-structure with developing pair* $(\mathbf{dev}, \mathbf{hol})$*. Then every small deformation of* **hol** *is the holonomy of a (G, X)-structure on M close to the initial one. Moreover, this (G, X)-structure is unique up to isotopy.*

If X is Riemannian, then the Hopf–Rinow theorem implies that all closed (G, X)-manifolds are complete. In contrast, if X is a flag variety of a semisimple Lie group G, then X is compact, hence the only complete (G, X)-manifolds are finitely covered by X. However, there are often many closed incomplete (G, X)-manifolds, obtained for instance as quotients of open domains in X. Between these extremes, there are many homogeneous spaces for which it is unknown whether every closed (G, X)-manifold is complete. A famous conjecture of Markus asks whether closed locally affine manifolds with a parallel volume form are complete [141]. The analoguous question for manifolds locally modelled on a reductive homogeneous space is also open.

Conjecture 2.1.7 (Reductive Markus Conjecture)**.** *Let X be a reductive homogeneous space and M a closed manifold locally modelled on X. Then M is a quotient of X.*

Even though we have chosen to focus specifically on compact quotients, the general theory of (G, X)-manifolds and the Reductive Markus conjecture are key in understanding deformations of such quotients (see Section 2.3.4). The reductive Markus conjecture was proved by Klingler for Lorentzian manifolds of constant curvature [106], building on the arguments of Carrière in the flat case [41]. In [186], we tried to push further these arguments and obtained a partial result towards the Markus conjecture for manifolds locally modelled on group spaces of rank 1.

Theorem 2.1.8. *Let H be a semisimple Lie group of rank 1 and Γ a discrete subgroup of $H \times H$. Assume Γ acts properly discontinuously and cocompactly on an open domain $U \subset H$. Then $U = H$.*

The (G, X)-manifolds obtained as quotients of open domains in X are sometimes called *Kleinian*, and Theorem 2.1.8 asserts that compact Kleinian $(H \times H, H)$-manifolds are complete. As a corollary, one obtains that completeness is a closed condition in the space of $(H \times H, H)$-structures on a closed manifold. Combined with an openness theorem of Guéritaud–Guichard–Kassel–Wienhard (Theorem 2.3.21), one concludes that it is impossible to deform continuously a complete $(H \times H, H)$-structure into an incomplete one.

2.2 Obstructions to compact quotients

Let us now review the various known obstructions to the existence of compact quotients of certain reductive homogeneous spaces. We propose to sort them into three categories: geometric obstructions, cohomological obstructions and dynamical obstructions.

2.2.1 Geometric obstructions

The geometric obstructions we discuss here relate to the fact that, in order to act properly discontinuously on G/H, a discrete subgroup of G has to move the compact subspace G^σ/H^σ "away" from itself.

The displacement of a maximal compact subspace by an element $g \in G$ is captured by the distance between the Cartan projection of g and those of H. For instance, gG^σ/H^σ intersects G^σ/H^σ in G/H if and only if there exists $h \in H$ such that

$$g = k_1 h k_2$$

for $k_1, k_2 \in G^\sigma$, which exactly means that

$$\mu(g) = \mu(h) \ .$$

As a first consequence of this remark, one obtains a general setting for the so called *Calabi–Markus phenomenon*:

Theorem 2.2.1 (Calabi–Markus phenomenon [39, 204, 107])**.** *If G and H have the same real rank, then any discrete subgroup of G acting properly discontinuously on G/H is finite.*

Indeed, when G and H have the same real rank, $\mu(H) = \mathfrak{a}_+$.

Corollary 2.2.2. *The pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q}$ does not admit a compact quotient when $p \leq q$.*

Elaborating on the previous computation, one obtains the useful properness criterion of Benoist–Kobayashi:

Lemma 2.2.3. *A discrete subgroup $\Gamma \subset G$ acts properly discontinuously on G/H if and only if for every $R > 0$, the set*

$$\{\gamma \in \Gamma \mid \exists h \in H, \|\mu(\gamma) - \mu(h)\| \leq R\}$$

is finite.

This criterion was exploited independently by Benoist and Kobayashi to obtain obstructions to the existence of compact Clifford–Klein forms. Kobayashi used the following obstruction:

Theorem 2.2.4 (Kobayashi, [108]). *Assume there exists a reductive subgroup H' of G such that $\mu(H') \subset \mu(H)$ and $\dim_+(G/H') < \dim_+(G/H)$. Then G/H des not admit a compact Clifford–Klein form.* [1]

Here is one application of this theorem:

Corollary 2.2.5 (Kobayashi, [108]). *For $d \geq 2$, the reductive homogeneous space $\mathrm{SL}(d, \mathbb{C})/\mathrm{SO}(d, \mathbb{C})$ does not admit a compact quotient.*

Benoist, on the other side, investigated further the structure of Cartan projections of a Zariski dense subgroup of G. He proved that these are asymptotic to a convex cone with non-empty interior, invariant under the opposition involution. As a consequence, he obtains the following

Theorem 2.2.6 (Benoist, [17]). *Assume that $\mu(H)$ contains the fixed points of the opposition involution. Then G/H does not have any compact quotient (unless it is itself compact).*

Corollary 2.2.7 (Benoist, [17]). *The space $\mathbb{H}^{2k+1,2k}$ $(k \geq 1)$ does not admit a compact quotient. The space $\mathrm{SL}(2k+1, \mathbb{R})/\mathrm{SL}(2k, \mathbb{R})$ $(k \geq 1)$ does not admit a compact quotient.*

2.2.2 Cohomologous obstructions

Our second family of obstructions uses vanishing results in cohomology to contradict the existence of compact quotients of certain G/H, whose volume would necessarily be non-zero. The first and main example is due to Kulkarni, who proved the following:

Theorem 2.2.8 (Kulkarni). *The space $\mathbb{H}^{p,q}$ does not admit a compact quotient when p and q are odd.*

Proof. The pseudo-Riemannian version of the Chern–Gauss–Bonnet formula together with the constant curvature of $\mathbb{H}^{p,q}$ imply that the volume of a compact quotient of $\mathbb{H}^{p,q}$ is proportional to its Euler characteristic when $p + q$ is even.

On the other side, the tangent space of every pseudo-Riemannian manifold of signature (p, q) admits a smooth decomposition as $V \oplus V^{\perp}$, where V is a positive definite subbundle of rank p. If p is odd, this implies the vanishing of the Euler characteristic, contradicting the non-vanishing of the volume. □

1. The contradiction here comes from the fact that the virtual cohomological dimension of a group Γ acting properly discontinuously and cocompactly on G/H must be $\dim_+(G/H)$. Our distinction between geometric and cohomological obstructions is thus somewhat artificial.

Kobayashi and Ono later gave the following interpretation of the previous obstruction. For a subgroup L of a Lie group G, let us denote by $\mathrm{H}^\bullet((G/L)^G)$ the cohomology of the complex of G-invariant forms on G/L. The projection $p_1 : G/H^\sigma \to G/H$ induces a homomorphism

$$p_1^* : \mathrm{H}^\bullet((G/H)^G) \to \mathrm{H}^\bullet((G/H^\sigma)^G) ,$$

and the G-invariant volume form $vol_{G/H}$ of G/H defines a class $[vol_{G/H}]$ in $\mathrm{H}^\bullet((G/H)^G)$.

Theorem 2.2.9 (Kobayashi–Ono [111]). *If $p_1^*[vol_{G/H}] = 0$, then G/H does not admit any compact quotients.*

These ideas were further developed independently by Morita [151] and myself [188]. Let us introduce the projection $p_2 : G/H^\sigma \to G/G^\sigma$, whose fibers are compact of dimension $\dim_-(G/H)$. The map p_2 induces a linear map

$$p_{2*} : \mathrm{H}^\bullet((G/H^\sigma)^G) \to \mathrm{H}^{\bullet - \dim_-(G/H)}((G/G^\sigma)^G)$$

obtained by "integrating along the fibers".

Theorem 2.2.10 (Tholozan [188]). *If $p_{2*}p_1^* vol_{G/H} = 0$, then G/H does not admit a compact quotient.*

The theorem follows from Theorem 2.4.12, which expresses the volume of $\Gamma\backslash G/H$ as the integral of the form $p_{2*}p_1^*vol_{G/H}$ against a cycle in $\mathrm{H}_{\dim_+(G/H)}(\Gamma)$. I developed several arguments to prove the vanishing of the above form in some cases. This turned out to give a rather powerful obstruction to the existence of compact Clifford–Klein forms. One of its striking consequences is the following:

Corollary 2.2.11 (Morita [151], Tholozan [188]). *The pseudo-Riemannian hyperbolic spaces $\mathbb{H}^{p,q}$ do not admit compact quotients when p is odd.*

Remark 2.2.12. Yosuke Morita proved this result using the projections $p_1' : G/T_H \to G/H$ and $p_2' : G/T_H \to G/T_G$, where T_H and T_G are respectively maximal tori of H^σ and G^σ. The two results are likely to be equivalent and have so far had the same consequences.

Let us finally mention a related obstruction by Benoist–Labourie:

Theorem 2.2.13 (Benoist–Labourie [21]). *Let G/H be a reductive homogeneous space such that the center of H contains a one parameter subgroup whose adjoint action is diagonalizable over $\mathbb{R}$. Then G/H does not admit a compact quotient.*

Their proof relies on the construction of a G-equivariant principal $\mathbb{R}$-bundle over G/H, the curvature of which defines a G-invariant closed 2-form ω. They prove that this form divides $vol_{G/H}$ in $H^\bullet(G/H^G)$. Though this form is not G-equivariantly exact, it factors to an exact form on every quotient of G/H because any principal $\mathbb{R}$-bundle admits a section.

2.2.3 Dynamical obstructions

To complete this overview of the existence problem for compact Clifford–Klein forms, let me briefly mention a third category of obstructions revolving around rigidity properties in homogeneous dynamics.

A first dynamical approach was developed in a series of papers by Zimmer, Labourie and Mozes [211, 123, 125]. Their strategy applies when the centralizer of H in G contains a simple group Z of real rank at least 2. The group Z acts on the right on a hypothetical compact quotient $\Gamma\backslash G/H$. Appliying Zimmer's cocycle rigidity [210] to the induced action on the principal H-bundle over $\Gamma\backslash G/H$ eventually leads to a contradiction in many cases. One of the achievements of this approach is the following:

Theorem 2.2.14 (Labourie–Zimmer [125])**.** *If* $2 \leq m \leq n-3$*, then the homogeneous space* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ *does not admit a compact quotient.*

In a similar direction, Shalom studied compact quotients of G/H where H has a large centralizer (though not necessarily of rank ≥ 2). His obstruction appears as a consequence of his deep work extending some higher rank rigidity results to Lie groups of rank one and their lattices. He obtains in particular the following

Theorem 2.2.15 (Shalom [176])**.** *If* $n \geq 4$*, then* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$ *does not admit a compact quotient.*

Shalom's approach is not unrelated to that of Margulis [139]. The latter shows that G/H does not admit any compact quotient when H is (G,K)-*tempered* – a sort of uniform Howe–Moore property for H in restriction to unitary representations of G. This theorem applies for instance when H is abelian by the Howe–Moore property of G. Margulis gives other interesting examples, among which the following:

Theorem 2.2.16. *Let* ι_n *be the irreducible representation of* $\mathrm{SL}(2,\mathbb{R})$ *of rank* n*. Then the homogeneous space* $\mathrm{SL}(n,\mathbb{R})/\iota_n(\mathrm{SL}(2,\mathbb{R}))$ *does not admit a compact quotient.*

2.3 Construction of compact quotients

We exhibited many obstructions to the existence of compact quotients of reductive homogeneous spaces. In comparison, there are few ways of constructing such quotients.

2.3.1 Standard quotients

The first source of examples are the standard quotients. Recall that a compact quotient $\Gamma\backslash G/H$ is called *standard* if Γ is virtually a uniform lattice in a connected Lie subgroup L of G.

Table 2.1 – List of standard triples (from [112])

G	H	L
$\mathrm{SO}(2d,2)$	$\mathrm{SO}(2d,1)$	$\mathrm{U}(d,1)$
$\mathrm{SU}(2d,2)$	$\mathrm{SU}(2d,1)$	$\mathrm{Sp}(d,1)$
$\mathrm{SO}(4d,4)$	$\mathrm{SO}(4d,3)$	$\mathrm{Sp}(d,1)$
$\mathrm{SO}(8,8)$	$\mathrm{SO}(8,7)$	$\mathrm{Spin}(8,1)$
$\mathrm{SO}(8,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{Spin}(7,1)$

The subgroup L must then act properly and cocompactly on G/H, and conversely, if L acts properly and cocompactly on G/H, then any uniform lattice in L gives a compact quotient. Finding such an L turns out to be an easier problem. For a reductive L, we have for instance the following characterization:

Proposition 2.3.1 (Kobayashi [107])**.** *A connected reductive subgroup L of G acts properly discontinuously on G/H if and only if $\mu(L) \cap \mu(H) = \{0\}$. If so, L acts cocompactly if and only if*

$$\dim_+(L) = \dim_+(G/H) \ .$$

If the conditions above are satisfied, we call (G, H, L) a *standard triple*. Note that standard triples have a symmetry: if L acts properly and cocompactly on G/H, then H acts properly and cocompactly on G/L.

The first example of a reductive standard triple was given by Kulkarni [116], who remarked that the group $\mathrm{U}(p, 1) \subset \mathrm{SO}(2p, 2)$ acts properly and transitively on $\mathrm{AdS}^{2p+1} = \mathrm{SO}(2p, 2)/\mathrm{SO}(2p, 1)$, thus proving the existence of closed anti-de Sitter manifolds in all odd dimensions. Kobayashi carried further the study of standard quotients. He gave an extended list of standard triples (see Table 2.3.1), from which one obtains for instance that $\mathbb{H}^{p,q}$ (with $p, q \geq 1$) admits standard quotients if and only if

$$(p, q) \in \{(k, 0), (2k, 1), (4k, 3), (8, 7)\} \ .$$

Let us finally note that all group manifolds admit standard quotients. Indeed the subgroup $H \times \{\mathrm{Id}\} \subset H \times H$ acts simply transitively on H (by left multiplication). The corresponding standard quotients are simply the left quotients $\Gamma\backslash H$, where Γ is a uniform lattice in H. Though these may seem somehow trivial, they will turn out quite interesting because they are the easiest to deform.

As we will see next, non-standard quotients, even when they exist, are closely related to standard ones. This motivates Kobayashi's space form conjecture:

Conjecture 2.3.2 (Kobayashi [107]). *If a reductive homogeneous space G/H admits a compact quotient, then it admits a standard one.*

This conjecture does not claim, however, that all compact quotients are standard. There are indeed two families of reductive homogeneous spaces that are known to admit non-standard compact quotients: the group spaces of rank 1 and the spaces $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$. Their common feature is the existence of deformations of standard quotients (while standard quotients of other reductive spaces are rigid).

2.3.2 Non-standard quotients: rank one group spaces

Goldman [74] was the first to realize that one could obtain non-standard quotients of group spaces by deforming standard ones. He remarked that torsion-free cocompact lattices of $\mathrm{PSL}(2,\mathbb{R}) \times \{\mathrm{Id}\}$ can be deformed non trivially into $\mathrm{PSL}(2,\mathbb{R}) \times A$ (where $A \subset \mathrm{PSL}(2,\mathbb{R})$ denotes the subgroup of diagonal matrices), and proved that such small deformations keep acting properly discontinuously on the group space $\mathrm{PSL}(2,\mathbb{R})$, which is isomorphic to AdS^3.

Remark 2.3.3. In contrast, standard quotients of AdS^{2d+1}, $d \geq 2$ are essentially rigid by a theorem of Raghunathan (see [96]).

Ghys [69] subsequently studied deformations of standard quotients of $\mathrm{SL}(2,\mathbb{C})$ and proved that they correspond locally to deformations of the underlying complex structure; while Kobayashi, pursuing the work of Kulkarni–Raymond [117] on anti-de Sitter 3-manifolds, initiated the more systematic study of compact quotients of group spaces of rank 1.

Let Γ be a uniform lattice in a semisimple Lie group H and ρ a homomorphism from Γ to H. We denote by Γ_ρ the *graph of* ρ in $H \times H$, i.e.

$$\Gamma_\rho = \{(\gamma, \rho(\gamma)), \gamma \in \Gamma\} \ .$$

When ρ is a deformation of the trivial representation (sending every element to the identity), the group Γ_ρ is a deformation of $\Gamma \times \{\mathrm{Id}\} \subset H \times H$.

Theorem 2.3.4 (Kobayashi [109]). *Let H be a semisimple Lie group of rank* 1 *with finite center and Γ' a discrete subgroup of $H \times H$ acting properly discontinuously on H. Then some finite index subgroup of Γ' has the form Γ_ρ for some uniform lattice $\Gamma \subset H$ and some $\rho : \Gamma \to H$.*

Remark 2.3.5. We insist here on the hypothesis that H has finite center because the group $H = \mathrm{SU}(d,1)$ has fundamental group $\mathbb{Z}$, so its universal cover has infinite center. Whether compact quotients of $\widetilde{H}$ are in fact covered by a finite cover of H is the so-called *level finiteness* problem. It was answered positively by Salein for $H = \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SU}(1,1)$ in his thesis [169]. To my knownledge, the level finiteness problem is unsolved for $\mathrm{SU}(d,1)$, $d \geq 2$, though Salein's arguments may generalize.

Remark 2.3.6. Theorem 2.3.4 does not generalize to higher rank. Indeed, if a reductive homogeneous space G/H admits a compact quotient $\Gamma\backslash G/H$, and if Λ is a uniform lattice in H, then $\Gamma \times \Lambda \subset G \times G$ acts properly discontinuously and cocompactly on the group space G.

The next step would be to understand, given a uniform lattice Γ in H, which representations $\rho : \Gamma \to H$ do give rise to compact quotients. This leads to the following definition:

Definition 2.3.7. A homomorphism ρ from a uniform lattice $\Gamma \subset H$ to H is called *admissible* if Γ_ρ acts properly discontinuously and cocompactly on on H.

For instance, the trivial representation

$$\begin{array}{rccc} \rho_{triv}: & \Gamma & \to & H \\ & \gamma & \mapsto & \mathbf{1}_H \end{array} .$$

is admissible, since $\Gamma_{\rho_{triv}} = \Gamma \times \{\mathrm{Id}_H\}$. Kobayashi, extending Ghys's result for $\mathrm{PSL}(2,\mathbb{C})$, proved that admissible representations contain a small neighbourhood of the trivial representation [110].

One thus obtains non-standard quotients of H as long as ρ_{triv} admits deformations that do not take values in a compact subgroup. Such deformations do not exist for $H = \mathrm{Sp}(d,1)$ or F_4^{-20}, according to the superrigidity theorem of Corlette [48]. In contrast, Kazhdan [101] and Millson [146] proved that $\mathrm{SU}(d,1)$ and $\mathrm{SO}(d,1)$ admit uniform lattices with non-trivial abelianization. For such a lattice, the trivial representation can be deformed into non-compact 1-parameter subgroups of H. Some lattices have an even richer deformation theory: one can show for instance that many uniform lattices in $\mathrm{SO}(d,1)$ surject onto a free group of rank at least 2, while Livné gave examples of uniform lattices in $\mathrm{SU}(2,1)$ that surject onto a surface group of genus at least 2 (hence also on a free group) [131]. For such lattices, the trivial representation can be deformed into a Zariski dense representation. It appears to be unknown whether such deformations exist in $\mathrm{SU}(d,1)$, $d \geq 3$.

These deformation results motivated the search for more precise admissibility criteria for representations of rank 1 lattices, which eventually lead to the construction of *exotic compact quotients* of rank 1 group manifolds, i.e. compact quotients that are *not* continuous deformations of standard ones.

The first such examples were constructed by Salein for $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}(2,1)$ [170] using a sufficient condition for properness that easily extends to other rank one Lie groups. Let Γ be a uniform lattice in H and ρ a morphism from Γ to H Let X denote the symmetric space of X.

Definition 2.3.8. The representation ρ is called *uniformly contracting* if there exists a ρ-equivariant map from X to itself which is λ-Lipschitz for some $\lambda < 1$.

Lemma 2.3.9 (Salein, [170]). *If ρ is uniformly contracting, then ρ is admissible.*

The converse to this lemma and its relation to Benoist–Kobayashi's properness criterion are discussed in Section 2.3.4.

Salein then constructed examples of uniform lattices in $\mathrm{PSL}(2,\mathbb{R})$ (isomorphic to surface groups) admitting uniformly contracting representations of non-zero Euler class. Since the Euler class is locally constant on the character variety, such representations are not continuous deformations of the trivial representation.

Similarly, Lakeland and Leininger [126] recently constructed pairs (Γ, ρ) where Γ is a uniform lattice in $\mathrm{SO}(3,1) \simeq \mathrm{PSL}(2,\mathbb{C})$ or $\mathrm{SO}(4,1)$ and ρ is a uniformly contracting representation of Γ with non-zero *volume*. Again, such representations cannot be continuously deformed to the trivial representation. Their construction exploits the existence of tilings of 3 and 4-dimensional hyperbolic spaces by right-angled polytopes, and does not seem to generalize to higher dimension.

Using again Salein's properness criterion, one can construct exotic compact quotients of $\mathrm{SU}(2,1) \simeq \mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$. Indeed, let Γ be a torsion-free uniform lattice in $\mathrm{SU}(2,1)$ such that $\Gamma\backslash\mathbb{H}^2_{\mathbb{C}}$ admits a non-constant holomorphic map f to a Riemann surface Σ of genus at least 2 (as constructed by Livné [131]). By the uniformization theorem, Σ is biholomorphic to $\Gamma'\backslash\mathbb{H}^1_{\mathbb{C}}$. Moreover, f induces a surjective morphism $f_* : \Gamma \to \Gamma'$ and lifts to a f_*-equivariant holomorphic map $\widetilde{f} : \mathbb{H}^2_{\mathbb{C}} \to \mathbb{H}^1_{\mathbb{C}}$. Using the Schwarz lemma, one shows that this map is 1-Lipschitz. Now, let $\rho : \Gamma' \to \mathrm{SU}(1,1) \subset \mathrm{SU}(2,1)$ be a contracting representation with non-zero Euler class (see Section 3.2.2 for its existence). Then $\rho \circ f_* : \Gamma \to \mathrm{SU}(2,1)$ is contracting and has non-zero *Toledo invariant*. Hence it is admissible and cannot be deformed to the trivial representation.

2.3.3 Non-standard quotients: $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$

The pseudo-Riemannian symmetric spaces $X_d = \mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ form the second family of reductive homogeneous spaces that are known to admit non-standard compact quotients.

The standard quotients of X_d have the form $\Gamma\backslash X_d$, where Γ is vitually a uniform lattice in $\mathrm{SO}(2d,1)$. It turns out that certain such lattices can be deformed into Zariski dense subgroups of $\mathrm{SO}(2d,2)$. Those deformations are the anti-de Sitter analog of certain deformations into $\mathrm{SO}(2d{+}1,1)$ introduced by Johnson and Millson in [92], which themselves generalize the well-known "bending deformations" of Fuchsian groups into $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{SO}(3,1)$.

To be slightly more precise, let Γ be a torsion-free uniform lattice in $\mathrm{SO}(2d,1)$ such that $\Gamma\backslash\mathbb{H}^{2d}$ admits a totally geodesic embedded hypersurface Σ. (Such Γ can be obtained for instance by taking congruence subgroups of arithmetic lattices.) Let i denote the inclusion of Γ into $\mathrm{SO}(2d,2)$.

This inclusion comes with a totally geodesic equivariant spacelike embedding of $\mathbb{H}^{2d}$ into AdS^{2d+1}. This embedding can now be "bent" equivariantly along the lifts of Σ to give a new piecewise totally geodesic embedding of $\mathbb{H}^{2d}$, which is ρ-equivariant for some deformation ρ of the inclusion. Such deformations typically have Zariski-dense image.

Kassel [97] proved in her thesis that, for ρ in a small neighbourhood of the inclusion, the group $\rho(\Gamma)$ keeps acting properly discontinuously and cocompactly on $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$. A stronger result was given by Guéritaud–Guichard–Kassel–Wienhard [80], building on the work of Barbot [13].

Barbot studied deformations of the Globally Hyperbolic Cauchy Compact (GHC) anti-de Sitter spacetimes associated to the above deformations. To avoid entering into the details of the theory, we will use the following definition:

Definition 2.3.10. A subgroup Γ of $\mathrm{SO}(d,2)$ is *convex-GHC* if Γ acts properly discontinuously and cocompactly on a complete convex spacelike hypersurface $\mathcal{H}$ in AdS^{d+1}.

Remark 2.3.11. This definition implies the existence of a maximal Γ-invariant domain of discontinuity $\Omega_\Gamma \subset \mathrm{AdS}^{d+1}$ whose quotient by Γ is homeomorphic to $(\Gamma\backslash\mathcal{H}) \times \mathbb{R}$. Those quotients are the *Globally Hyperbolic Cauchy compact* AdS spacetimes mentionned in the introduction.

Barbot proved the following very strong stability theorem for convex-GHC groups:

Theorem 2.3.12 (Barbot [13]). *Let* Γ *be a uniform lattice in* $\mathrm{SO}(d,1)$ *and* $\rho : \Gamma \to \mathrm{SO}(d,2)$ *any continuous deformation of the inclusion. Then* ρ *is injective and* $\rho(\Gamma)$ *is convex-GHC.*

He also proved with Mérigot [14] that convex-GHC groups are Anosov with respect to the parabolic subgroup of $\mathrm{SO}(d,2)$ preserving an isotropic line in $\mathbb{R}^{d,2}$. As a corollary, one obtains the following:

Theorem 2.3.13 (Guéritaud–Guichard–Kassel–Wienhard, [80]). *Let* Γ *be a convex-GHC subgroup of* $\mathrm{SO}(2k,2)$. *Then* Γ *acts properly discontinuously and cocompactly on* $\mathrm{SO}(2k,2)/\mathrm{U}(k,1)$. *In particular, if* Γ *is a uniform lattice in* $\mathrm{SO}(2k,1)$*, then any continuous deformation of* Γ *into* $\mathrm{SO}(2k,2)$ *acts properly discontinuously and cocompactly on* $\mathrm{SO}(2k,2)/\mathrm{U}(k,1)$.

Barbot conjectured that every convex-GHC subgroup of $\mathrm{SO}(k,2)$ is virtually a deformation of a uniform lattice in $\mathrm{SO}(k,1)$. This was recently disproved by Lee and Marquis [128]: for $4 \leq k \leq 8$, they exihibited examples of convex-GHC Coxeter groups which are not isomorphic to hyperbolic lattices. We construct further examples in a forthcoming work with Jean-Marc Schlenker and Daniel Monclair [149] (see Section 2.5.1). These examples imply the existence of exotic quotients of $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$.

2.3.4 Openness and Sharpness

The examples described in the previous section show that compact quotients of reductive homogeneous spaces sometimes come in continuous families. It is thus interesting to try to describe as precisely as possible the topology of those families.

To make the question more concrete, let us fix a finitely generated torsion-free group Γ and consider the space

$$\widetilde{\mathrm{PDC}}(\Gamma, G/H) = \{\rho : \Gamma \to G \mid \rho \text{ acts prop. disc. and cocomp. on } G/H\} \ .$$

Note that $\widetilde{\mathrm{PDC}}(\Gamma, G/H)$ is trivially invariant under conjugation by G. We denote by $\mathrm{PDC}(\Gamma, G/H)$ its quotient under conjugation. Note that the description of $\mathrm{PDC}(\Gamma, G/H)$ is deeply related (though not always equivalent) to the description of the space of compact quotients of G/H homeomorphic to a given manifold.

A first natural question is whether the space $\widetilde{\mathrm{PDC}}(\Gamma, G/H)$ is open in $\mathrm{Hom}(\Gamma, G)$. This is conjecturally always the case:

Conjecture 2.3.14 (Openness conjecture)**.** *Let Γ be a finitely generated group and G/H a reductive homogeneous space. Then the set $\widetilde{\mathrm{PDC}}(\Gamma, G/H)$ is open in $\mathrm{Hom}(\Gamma, G)$.*

Recall that the Ehresmann–Thurston principle (Theorem 2.1.6) gives the following weaker statement: if $\rho \in \mathrm{PDC}(\Gamma, G/H)$ and $M = \rho(\Gamma)\backslash G/H$ is the corresponding compact quotient of G/H, then every small enough deformation ρ' of ρ is still the holonomy of a $(G, G/H)$-structure on M. If the reductive Markus conjecture (Conjecture 2.1.7) were true, this (G, X)-manifold ought to be complete, hence isomorphic to the compact quotient $\rho'(\Gamma)\backslash X$. In conclusion:

Proposition 2.3.15. *The reductive Markus conjecture implies the openness conjecture.*

While the reductive Markus conjecture is far from being resolved, the openness conjecture 2.3.14 is known to hold in all the previously discussed examples, where it is strongly related to the theory of Anosov subgroups via the following *sharpness conjecture*.

Let $|\cdot|$ denote some word metric on Γ associated to a finite system of generators. The following definition is adapted from Kobayashi.

Definition 2.3.16. We say that $\rho : \Gamma \to G$ is a *sharp embedding* with respect to H if there exists $C > 1$ such that for all $\gamma \in \Gamma$,

$$\inf_{h \in H} \|\mu(\rho(\gamma)) - \mu(h)\| \geq \frac{1}{C}|\gamma| - C \ .$$

More informally, an embedding of Γ is sharp with respect to H if the distance of its Cartan projections to those of H increases linearly with the word-length. Note the resemblance of the sharpness condition with the Anosov property (see Definition 1.3.9).

Remark 2.3.17. The above definition differs from the original definition of Kobayashi in that it includes the assumption that ρ is a quasi-isometric embedding (i.e. the Cartan projections grow linearly with the word length).

If ρ is a sharp embedding, its kernel is finite and its image satisfies Benoist–Kobayashi's properness criterion. Kassel and Kobayashi conjectured a converse to this remark for cocompact actions.

Conjecture 2.3.18 (Sharpness conjecture [98])**.** *Every $\rho \in \widetilde{\mathrm{PDC}}(\Gamma, G/H)$ is a sharp embedding with respect to H.*

Let us now discuss these two conjectures in the situations where the existence of non-rigid compact quotients is known.

Group spaces of rank 1

Let H be a Lie group of rank 1. Recall that the Cartan projection μ can be seen directly as a map from H to $\mathbb{R}_+$. Let Γ be a uniform lattice in H and ρ a morphism from $\Gamma \to H$.

Proposition 2.3.19. *The group $\Gamma_\rho \subset H \times H$ is sharply embedded with respect to $\Delta(H)$ if and only if there exists $C > 1$ such that*

$$\mu(\rho(\gamma)) \leq C\mu(\gamma) + C \ .$$

Interpreting the Cartan projection as a distance in the symmetric space, one easily deduces that Γ_ρ is sharply embedded when ρ is uniformly contracting, which proves Salein's properness criterion.

In her thesis, Kassel proved a converse to Salein's properness criterion for the group space $H = \mathrm{PSL}(2, \mathbb{R})$, namely that admissible representations of a lattice are uniformly contracted. This was extended to $H = \mathrm{SO}(d, 1)$ in [81], thus proving both the sharpness and openness conjecture in that case.

Theorem 2.3.20 (Guéritaud–Kassel [81])**.** *Let Γ be a uniform lattice in $\mathrm{SO}(d, 1)$ and ρ a morphism from Γ into $\mathrm{SO}(d, 1)$. Then the following propositions are equivalent:*

(*i*) *Γ_ρ acts properly discontinuously (and cocompactly) on $\mathrm{SO}(d, 1)$,*
(*ii*) *Γ_ρ is sharply embedded with respect to $\Delta(\mathrm{SO}(d, 1))$,*
(*iii*) *ρ is uniformly contracting.*

In particular, the openness conjecture holds for the group space $\mathrm{SO}(d, 1)$.

This result was partially generalized to all rank one Lie groups by Guéritaud–Guichard–Kassel–Wienhard. In [80], they prove the sharpness conjecture in that setting and relate it to some Anosov property, thus showing that it implies the openness conjecture. The novelty is the case of $\mathrm{SU}(d, 1)$.

Theorem 2.3.21 (Guéritaud–Guichard–Kassel–Wienhard [80])**.** *Let* Γ *be a uniform lattice in* $\mathrm{SU}(d,1)$ *and* ρ *an admissible morphism from* Γ *into* $\mathrm{SU}(d,1)$*. Then* Γ_ρ *is sharply embedded with respect to* $\Delta(\mathrm{SU}(d,1))$*. Moreover, the set of admissible morphisms is open in* $\mathrm{Hom}(\Gamma, \mathrm{SU}(d,1))$*.*

Remark 2.3.22. While Guéritaud–Guichard–Kassel–Wienhard's work could also apply to the other rank one Lie groups $\mathrm{Sp}(d,1)$ and F_4^{-20}, the conclusion there follows somewhat trivially from Corlette's superrigidity theorem [48].

Interestingly, the equivalence between properties (*ii*) and (*iii*) of Theorem 2.3.20 is not known for $\mathrm{SU}(d,1)$, and the methods of [81] do not seem to generalize (see Section 2.4.1).

The space $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$

The sharpness conjecture is still open for compact quotients of $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$. For discrete subgroups of $\mathrm{SO}(2d,2)$, sharpness with respect to $\mathrm{U}(d,1)$ is equivalent to the Anosov property with respect to the stabilizer of an isotropic line, from which one deduces:

Proposition 2.3.23. *Let* Γ *be a discrete subgroup of* $\mathrm{SO}(2d,2)$ *acting properly discontinuously on* $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$*. If* Γ *is sharp with respect to* $\mathrm{U}(d,1)$*, then* Γ *is convex-GHC.*

By Theorem 2.3.13, sharp quotients are both open and closed in the character variety, and proving the sharpness conjecture in this setting is equivalent to proving that all the compact quotients of $\mathrm{SO}(2k,2)/\mathrm{U}(k,1)$ come from convex-GHC groups.

2.3.5 The moduli space of compact quotients of AdS^3

Understanding the stability of compact quotients of G/H under deformation is a first step towards the topological and geometric description of the space $\mathrm{PDC}(\Gamma, G/H)$.

To go further, however, one would need a good understanding of the representation variety $\widehat{\mathfrak{X}}(\Gamma, G)$, which is inaccessible in most cases. The examples studied so far typically involve representations of $\mathrm{SO}(d,1)$ or $\mathrm{SU}(d,1)$-lattices, which are far from being well-understood beyond the case of $\mathrm{SO}(2,1) \simeq \mathrm{SU}(1,1) \simeq \mathrm{PSL}(2,\mathbb{R})$.

In contrast, many tools have been developed to study representations of surface groups. We present them in more details in the third chapter of this memoir. In my thesis, I applied some of those tools in order to describe the space of compact quotients of the group space $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{AdS}^3$.

Recall first that, up to finite index, the groups acting properly discontinuously and cocompactly on the group space $\mathrm{PSL}(2,\mathbb{R})$ are torsion-free uniform lattices in $\mathrm{PSL}(2,\mathbb{R})$, and are thus isomorphic to fundamental groups of closed surfaces of genus at least 2.

Let thus Γ be the fundamental group of a closed oriented surface Σ of genus at least 2, and let (j,ρ) be a pair of representations of Γ into $\mathrm{PSL}(2,\mathbb{R})$ such that j is *Fuchsian* (i.e. j identifies Γ with a lattice in $\mathrm{PSL}(2,\mathbb{R})$). We will say that j *dominates* ρ, or that (j,ρ) is a *dominating pair* if there exists a (j,ρ)-equivariant contracting map from $\mathbb{H}^2$ to $\mathbb{H}^2$. By the work of Kassel mentionned previously (see Theorem 2.3.20), describing the space of all compact quotients of AdS^3 boils down to describing the space of dominating pairs up to conjugation.

The space of discrete and faithful representations of Γ into $\mathrm{PSL}(2,\mathbb{R})$ modulo conjugation has two components $\mathcal{T}_\pm(\Sigma)$ which identify, via Poincaré's uniformization theorem, to the Teichmüller spaces of $\Sigma_\pm$ (i.e. Σ with both possible orientations). Each component is diffeomorphic to $\mathbb{R}^{6g-6}$, where g is the genus of Σ. Note also that, if (j,ρ) is a dominating pair, then ρ cannot be discrete and faithful (otherwise, $j(\Gamma)\backslash\mathbb{H}^2$ and $\rho(\Gamma)\backslash\mathbb{H}^2$ would have the same volume, which would contradict the existence of a contracting equivariant map).

In [189], I proved the following theorem:

Theorem 2.3.24 (See also Theorem 3.2.21)**.** *Let ρ be a representation of Γ into $\mathrm{PSL}(2,\mathbb{R})$. If ρ is not discrete and faithful, then the open set*

$$\{[j]\in\mathcal{T}_+(\Sigma)\mid j \text{ dominates } \rho\}\subset\mathcal{T}_+(\Sigma)$$

is homeomorphic to $\mathbb{R}^{6g-6}$.

This built on previous work with Bertrand Deroin [55] proving the non-emptiness of the set of dominating representations, a result obtained independently by Guéritaud–Kassel–Wolf [82]. We will come back on those results and their generalization in Section 3.2.2.

Theorem 2.3.24, combined with results of Hitchin on the topology of the $\mathrm{PSL}(2,\mathbb{R})$-character variety of a surface group (Theorem 3.1.9), allow for a complete topological description of the space $\mathrm{PDC}(\Gamma,\mathrm{AdS}^3)$. In particular, its connected components are classified by the *Euler classes* of j and ρ.

Interestingly, the topology of those quotients depends on these Euler classes. To be more precise, if (j,ρ) is an admissible pair then the quotient

$$j\times\rho(\Gamma)\backslash\mathrm{AdS}^3 \tag{2.1}$$

is diffeomorphic to a circle bundle over Σ of Euler class

$$\mathbf{eu}(j)-\mathbf{eu}(\rho)\ .$$

In my thesis I also describe accurately the deformation space of anti-de Sitter structures on a fixed circle bundle over Σ [187, Section 4.4]. This space can have several connected components, some of which are obtained by taking finite cyclic covers or cyclic quotients of the quotients of the form (2.1).

2.4 Geometry of compact quotients

I will conclude this presentation of compact quotients of reductive homogeneous spaces by investigating their topology and geometry. I will first present a conjectural picture of those quotients which is known to hold in many cases, then I will explain how this conjecture inspired my work on volumes of those compact quotients.

2.4.1 A conjectural picture

Let G/H be a reductive homogeneous space and σ a Cartan involution of G preserving H. Recall that G/H admits a retraction to the compact negative definite subspace G^σ/H^σ. In [188], I conjectured the following:

Conjecture 2.4.1 (Fibration conjecture)**.** *Let Γ be a discrete subgroup of G acting properly discontinuously on G/H. Then, up to taking a finite index subgroup:*

(1) Γ is the fundamental group of a closed aspherical manifold M of dimension $\dim_+(G/H)$,

(2) There exists a smooth Γ-invariant fibration from G/H to $\widetilde{M}$ whose fibers have the form $g\, G^\sigma/H^\sigma$ for some $g \in G$.

We call a fibration $G/H \to \widetilde{M}$ satisfying (2) (or the induced fibration $\Gamma\backslash G/H \to M$) a *geometric fibration*.

Remark 2.4.2. In all the examples discussed below, the "up to finite cover" assumption is only used to pass to a torsion-free subgroup, which is a necessary condition for Γ to satisfy (1). One could formulate the stronger conjecture that $\Gamma\backslash G/H$ always admits a geometric fibration over a closed *negatively curved Riemannian orbifold* with fundamental group Γ.

Here I want to argue in favour of this conjecture by pointing out that it holds in most known cases. Let us start by mentioning the case of standard quotients.

Proposition 2.4.3. *The fibration conjecture holds for standard quotients.*

Proof. Let L be a connected Lie subgroup of G acting properly and cocompactly on G/H. By conjugating L one reduces to the case where L in σ-invariant. Using Benoist–Kobayashi's properness criterion, one shows that

$$\{g \in L \mid gG^\sigma/H^\sigma \cap G^\sigma/H^\sigma \neq \emptyset\} = L^\sigma \ ,$$

while the cocompactness of the action of L implies that

$$L \cdot (G^\sigma/H^\sigma) = G/H \ .$$

This gives a well-defined smooth L-equivariant fibration $\pi : G/H \to L/L^\sigma$ such that

$$\pi^{-1}(gL^\sigma) = g(G^\sigma/H^\sigma) \ .$$

Now, any torsion-free uniform lattice Γ in L is the fundamental group of the closed negatively curved manifold $\Gamma\backslash L/L^\sigma$, and π is the desired equivariant geometric fibration. $\square$

One can furthermore show that the conjecture remains true under small deformations:

Proposition 2.4.4. *Assume* $\Gamma\backslash G/H$ *satisfies the fibration conjecture. Then there exists a small neigbourhood* U *of the inclusion* $\Gamma \hookrightarrow G$ *in* $\mathrm{Hom}(\Gamma, G) \cap \widetilde{\mathrm{PDC}}(\Gamma, G/H)$ *such that for all* $\rho \in U$*, the quotient* $\rho(\Gamma)\backslash G/H$ *satisfies the fibration conjecture.*

These are the first hints that the conjecture might hold in general. We now turn to more concrete situations of which we have a better understanding.

Group spaces of rank 1

For compact quotients of the group space $\mathrm{SO}(d, 1)$, the fibration conjecture was proved by Guéritaud–Kassel as a consequence of Theorem 2.3.20.

Theorem 2.4.5 (Guéritaud–Kassel [81])**.** *Compact quotients of the group space* $\mathrm{SO}(d, 1)$ *satisfy the fibration conjecture.*

Proof. By Theorem 2.3.20, after taking a finite cover, one is reduced to study quotients of the form $\Gamma_\rho\backslash\mathrm{SO}(d, 1)$, where Γ is a torsion-free uniform lattice in $\mathrm{SO}(d, 1)$ and $\rho : \Gamma \to \mathrm{SO}(d, 1)$ a contracting representation.

Let $f : \mathbb{H}^d \to \mathbb{H}^d$ be a smooth ρ-equivariant contracting map, and define

$$\pi : \mathrm{SO}(d, 1) \to \mathbb{H}^d$$

which associates to g the unique fixed point of $x \mapsto g \cdot f(x)$.

The smoothness and contraction property of f imply that π is a smooth submersion. One easily verifies that π is equivariant with respect to the actions of Γ_ρ on $\mathrm{SO}(d, 1)$ and Γ on $\mathbb{H}^d$. Finally, the fiber at a point x has the form $\mathrm{Stab}(x)g$, where $\mathrm{Stab}(x)$ is the compact subgroup of $\mathrm{SO}(d, 1)$ fixing x. The submersion π is thus a geometric fibration. $\square$

Note that the same proof works for compact quotients of $\Gamma_\rho\backslash\mathrm{SU}(d, 1)$ when $\rho : \Gamma \to \mathrm{SU}(d, 1)$ is a contracting representation. In fact, one can prove that, for an admissible representation ρ of a uniform lattice $\Gamma \subset \mathrm{SU}(d, 1)$, the quotient $\Gamma_\rho\backslash\mathrm{SU}(d, 1)$ admits a geometric fibration if and only if ρ is contracting. The fibration conjecture can thus be reformulated as

Conjecture 2.4.6. *Let* Γ *be a uniform lattice in* $\mathrm{SU}(d,1)$ *and* $\rho : \Gamma \to \mathrm{SU}(d,1)$ *an admissible representation. Then* ρ *is contracting.*

The sharpness of compact quotients of $\mathrm{SU}(d,1)$ (Theorem 2.3.21), implies that if $\rho : \Gamma \to \mathrm{SU}(d,1)$, and $f : \mathbb{H}^d_{\mathbb{C}} \to \mathbb{H}^d_{\mathbb{C}}$ is a continuous ρ-equivariant map, then there exists $\lambda < 1$ and $C > 0$ such that

$$d(f(x), f(y)) < \lambda d(x, y)$$

for all x, y such that $d(x, y) > C$. Conjecture 2.4.6 asks whether this large scale contraction property is true uniformly for some f. The tools used by Guéritaud–Kassel in the $\mathrm{SO}(d,1)$ case (namely, a Kirzbraum–Valentine extension theorem for Lipschitz maps of the hyperbolic space) do not generalize to $\mathrm{SU}(d,1)$ because of the variable curvature of $\mathbb{H}^d_{\mathbb{C}}$.

However, there is hope to deduce Conjecture 2.4.6 from rigidity theorems for equivariant harmonic maps: given a (reductive) representation $\rho : \Gamma \to \mathrm{SU}(d,1)$ of a uniform lattice Γ, there exists a ρ-equivariant harmonic map $f : \mathbb{H}^d_{\mathbb{C}} \to \mathbb{H}^d_{\mathbb{C}}$. A theorem of Siu implies that this map is either holomorphic or locally factors trough a holomorphic map to a curve. In the first case, this map must be 1-Lipschitz by the Schwarz lemma, and one can prove that it is contracting if ρ is admissible. One might hope to reduce the latter case to a statement about surface group representations, for which we have many tools at our disposal (see Chapter 3).

The particular case of anti-de Sitter 3-manifolds

By the isomorphism $\mathrm{SO}_0(2,1) \simeq \mathrm{AdS}^3$, the fibration conjecture holds for compact quotients of AdS^3, as a particular case of Theorem 2.4.5. Here, compact subspaces are exactly the timelike geodesics and Theorem 2.4.5 admits the following remarkable formulation:

Corollary 2.4.7. *every closed anti-de Sitter* 3*-manifold admits a foliation by timelike geodesics.*

Stated this way, the result appears of analytic nature. Every time-oriented Lorentzian manifold admits unit timelike vector fields, and one wishes to find such a vector field X satisfying the equation

$$\nabla_X X = 0 \ ,$$

where ∇ denotes the Levi–Civita connection. It is tempting to try to imagine an analytic approach to this question, which would extend beyond the case of closed AdS 3-manifolds. One could wonder for instance whether Corollary 2.4.7 extends to higher dimension, or to Lorentzian manifolds which are “close to” being anti-de Sitter.

Note that timelike geodesic foliations, when they exist, are not necessarily unique. For the closed anti-de Sitter 3-manifold $j \times \rho(\Gamma)\backslash \mathrm{AdS}^3$, they are in bijection with the (j,ρ)-equivariant contracting maps from $\mathbb{H}^2$ to $\mathbb{H}^2$. On the other side, one might hope that an analytic approach provides a canonical choice of geodesic foliation, solution of some variational problem.

From my work [189], one can deduce a canonical choice of a (j,ρ)-equivariant contracting map, whose graph is a maximal spacelike surface in $\mathbb{H}^2 \times \mathbb{H}^2$ endowed with the pseudo-Riemannian metric $g_{\mathbb{H}^2} \oplus -g_{\mathbb{H}^2}$ (see Section 3.2.2). It would be interesting to characterize this map in terms of the associated timelike foliation, and see in particular if it minimizes some functional on the space of unit timelike vector fields.

The spaces $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$

In the work in preparation [149], we prove that the Fibration conjecture holds for sharp compact quotients of $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ (see Section 2.5.1).

2.4.2 Volume

Let $vol_{G/H}$ denote the G-invariant volume form on G/H associated to the pseudo-Riemannian metric. This form factors to every compact quotient of G/H and allows to define its *volume*:

$$\mathbf{Vol}(\Gamma\backslash G/H) = \left| \int_{\Gamma\backslash G/H} vol_{G/H} \right| .$$

In [191] and [188], I initiated a systematic study of volumes of compact reductive Clifford–Klein forms, driven by the following questions:

Question 2.4.8. *Is* $\mathbf{Vol}(\Gamma\backslash G/H)$ *rigid (i.e. constant under continuous deformations of* Γ*)?*

Question 2.4.9. *Is* $\mathbf{Vol}(\Gamma\backslash G/H)$ *rational (i.e. a rational multiple of a constant depending on* G/H*)?*

Question 2.4.10. *Does* $\mathbf{Vol}(\Gamma\backslash G/H)$ *have a cohomological interpretation?*

These three questions are intimately related. Obviously, the rationality of the volumes of compact quotients of G/H implies their rigidity. On the other side, rationality typically follows from interpreting this volume as a characteristic class via Chern–Weil theory.

To be more precise, note that the principal H-bundle $G \to G/H$ admits a G-invariant connection. Let Ω denote its curvature form. Assume there exists a H-invariant polynomial Q on $\mathfrak{h}$ such that

$$vol_{G/H} = Q(\Omega) .$$

Then, up to scaling $vol_{G/H}$ and ajusting Q, one can assume that $CW_H(Q)$ belongs to $\mathrm{H}^\bullet(BH, \mathbb{Z})$, where CW_H is the Chern–Weil homomorphism. After quotienting by Γ, we obtain that

$$\begin{aligned} \mathbf{Vol}(\Gamma\backslash G/H) &= \left| \int_{\Gamma\backslash G/H} Q(\Omega) \right| \\ &= \left| \int_{\Gamma\backslash G/H} f^* CW_H(Q) \right| \\ &\in \mathbb{Z}_{>0} \ , \end{aligned}$$

where f denotes the classifying map for the principal bundle $\Gamma\backslash G \to \Gamma\backslash G/H$.

A prototypical example is the hyperbolic plane, whose volume form is proportional to the curvature form of its tangent bundle with the Poincaré metric. One deduces the Gauss–Bonnet formula, namely that $-\frac{1}{2\pi}\mathbf{Vol}(\Gamma\backslash\mathbb{H}^2)$ equals the Euler class of the tangent bundle to $\Gamma\backslash\mathbb{H}^2$, i.e. the Euler characteristic of $\Gamma\backslash\mathbb{H}^2$. This proves the rationality and rigidity of the volume of closed hyperbolic surfaces.

When G/H is *symmetric* (i.e. H is the set of fixed points of an involution of G), a theorem of Cartan asserts that $vol_{G/H}$ is a Chern–Weil form if and only if G and H have the same complex rank. Of course $vol_{G/H}$ cannot be a Chern–Weil form if G/H has odd dimension. In particular, this argument does not give the rationality of the volume for

— compact quotients of $\mathbb{H}^{p,q}$, $p+q$ odd.

— compact quotients of (non compact) group spaces.

In my thesis, I obtained the following volume formula for closed anti-de Sitter 3-manifolds:

Theorem 2.4.11. *Let Γ be the fundamental group of a closed surface and (j,ρ) a contracting pair of representations from Γ to $\mathrm{PSL}(2,\mathbb{R})$. Then*

$$\mathbf{Vol}(j \times \rho(\Gamma)\backslash \mathrm{PSL}(2,\mathbb{R})) = \frac{\pi^2}{2} |\mathbf{eu}(j) + \mathbf{eu}(\rho)| \ ,$$

where **eu** *denotes the Euler class of a representation.*

This rationality of volumes of closed anti-de Sitter 3-manifolds is in stark contrast with the behaviour of the Riemannian analog, namely volumes of closed hyperbolic 3-manifolds.

Theorem 2.4.11 follows from a rather explicit differential geometric computation which consists in integrating the volume form along the fibers of a geometric fibration. It was obtained independently by Alessandrini and Li in [7]. In [191], I generalized it to (not necessarily compact) quotients of the group space $\mathrm{SO}_0(d,1)$.

In [188], I gave another interpretation of Theorem 2.4.11 that extends to a broader context. Let $\Gamma\backslash G/H$ be a compact reductive Clifford–Klein form and assume without loss of generality that Γ is torsion-free. Denote as previously p_1 and p_2 the respective projections from G/H^σ to G/H and G/G^σ. These projections factor through the proper action of Γ. The map p_1 has contractible fibers, hence it is a homotopy equivalence. The map p_2, on the other side, is a fibration with compact fibers of dimension p. A classical spectral sequence argument then shows that

$$\mathrm{H}_p(\Gamma) = \mathrm{H}_p(\Gamma\backslash G/G^\sigma) \simeq \mathbb{Z} .$$

It is thus generated by a an element $[\Gamma]$, which can be seen as a cycle in the classifying space $\Gamma\backslash G/G^\sigma$.

Theorem 2.4.12. *We have*

$$\mathbf{Vol}(\Gamma\backslash G/H) = \int_{[\Gamma]} p_{2*}p_1^*\omega_{G/H} .$$

Remark 2.4.13. When the form $p_{2*}p_1^*\omega_{G/H}$ vanishes, this contradicts the very existence of compact quotients of G/H. This proves Theorem 2.2.10.

To explain Theorem 2.4.12, let us interpretate G/G^σ as the space of translates of G^σ/H^σ in G/H and G/H^σ as the spaces of pairs (x, V) where V is a translate of G^σ/H^σ in G/H and x is a point in V. With these interpretations, we have $p_1(x, V) = x$ and $p_2(x, V) = V$. Recall that $p_{2*} : \Omega^\bullet(G/H^\sigma) \to \Omega^{\bullet - q}(G/G^\sigma)$ is the "integration over the fibers", characterised by the relation

$$\int_M p_{2*}\alpha = \int_{p_2^{-1}(M)} \alpha .$$

Assume first that $\Gamma\backslash G/H$ admits a geometric fibration π over a closed manifold M of dimension p. This fibration gives an embedding $i : M \to \Gamma\backslash G/G^\sigma$ (sending a point in M to its fiber) with the property that

$$p_1 : p_2^{-1}(i(M)) \to \Gamma\backslash G/H$$

is a diffeomorphism. We thus have

$$\begin{aligned} \mathbf{Vol}(\Gamma\backslash G/H) &= \int_{p_2^{-1}(i(M))} p_1^* vol_{G/H} \\ &= \int_{i(M)} p_{2*}p_1^*\omega_{G/H} . \end{aligned}$$

While a geometric fibration may not exist in general, it always exists "up to homology". To be more precise, let s be a continuous section of the fibration $p_1 : \Gamma\backslash G/H^\sigma \to \Gamma\backslash G/H$ (which exists because the fibers of p_1 are contractible). Applying the Leray–Serre spectral sequence to the homology of the fibration $p_2 : \Gamma\backslash G/H^\sigma \to \Gamma\backslash G/G^\sigma$, we show

Lemma 2.4.14. *Let* $[\Gamma]$ *be a cycle in* $\Gamma\backslash G/G^\sigma$ *generating* $H_p(\Gamma,\mathbb{Z})$*. Then* $p_2^{-1}([\Gamma])$ *is homologuous to* $s(\Gamma\backslash G/H)$ *in* $\Gamma\backslash G/H^\sigma$*.*

The previous computation is then easily adapted to prove Theorem 2.4.12.

Using the fact that the form $p_{2*}p_1^*\omega_{G/H}$ is in some sense "Poincaré dual" to the symmetric subspace $H/H^\sigma \subset G/G^\sigma$ (this statement can be made precise by considering the compact dual symmetric spaces) together with results of Cartan on the (invariant) cohomology of symmetric spaces, I could characterize when the form $p_{2*}p_1^*\omega_{G/H}$ is a Chern–Weil form. Define the complex rank of a semisimple Lie group G as the real rank of its complexification:

$$\mathbf{rk}_\mathbb{C}(G) = \mathbf{rk}(\mathrm{Aut}(\mathfrak{g}\otimes\mathbb{C}))\ .$$

Theorem 2.4.15. *If* $\omega_{G/H}$ *is a non-vanishing Chern–Weil form on* G/G^σ *then*

$$\mathbf{rk}_\mathbb{C}(G) - \mathbf{rk}_\mathbb{C}(G^\sigma) = \mathbf{rk}_\mathbb{C}(H) - \mathbf{rk}_\mathbb{C}(H^\sigma)\ . \tag{2.2}$$

Conversely, if Equality 2.2 holds, then $p_{2*}p_1^*\omega_{G/H}$ *is a (possibly vanishing) Chern–Weil form.*

This proves the rationality of the volume of compact quotients of many reductive homogeneous spaces, among which:

— The pseudo-hyperbolic spaces $\mathbb{H}^{p,q}$ for p even,

— The group spaces H, where H is a Lie group of Hermitian type.

For group spaces of rank 1, I characterized more explicitly the form $p_{2*}p_1^*\omega_{G/H}$. Combined with Kobayashi's Theorem 2.3.4, one obtains a precise formula for the volume of compact quotients. Let Γ be a uniform lattice in $H = \mathrm{SO}_0(d,1)$ or $\mathrm{SU}(d,1)$ and $\rho : \Gamma \to H$ an admissible representation. Recall that, given an H-invariant form α on H/H^σ, we denote by $\rho^*\alpha$ the form on $\Gamma\backslash H/H^\sigma$ obtained by pulling-back α by any smooth ρ-equivariant map, and factoring through Γ (cf Chapter 1, Section 1.2.3).

Theorem 2.4.16. *For* $H = \mathrm{SO}(d,1)$*, let* $vol_{\mathbb{H}^d}$ *denote an invariant volume form on* $H/H^\sigma = \mathbb{H}^d$*. Then we have*

$$\mathbf{Vol}(\Gamma_\rho\backslash H) = A_d\left|\int_{\Gamma\backslash\mathbb{H}^d} vol_{\mathbb{H}^d} + (-1)^d\int_{\Gamma\backslash\mathbb{H}^d}\rho^* vol_{\mathbb{H}^d}\right|\ .$$

For $H = \mathrm{SU}(d,1)$*, let* α *denote an invariant Kähler form on* $H/H^\sigma = \mathbb{H}^d_\mathbb{C}$*. Then we have*

$$\mathbf{Vol}(\Gamma_\rho\backslash H) = B_d\left|\sum_{k=0}^d \int_{\Gamma\backslash\mathbb{H}^d_\mathbb{C}} \alpha^k\wedge\rho^*\alpha^{d-k}\right|\ .$$

The constants A_d and B_d in the above theorem can be made explicit if we carefully normalize the various forms.

Remark 2.4.17. While the case of $\mathrm{SO}(d,1)$ was already treated in [191] using Guéritaud–Kassel's fibration, Theorem 2.4.12 gives it a more conceptual approach. For $d = 2$, we recover Theorem 2.4.11.

We could also apply Theorem 2.4.12 to compute the volume of sharp compact quotients of $\mathrm{SO}(2p,2)/\mathrm{U}(p,1)$ (see Section 2.5.1).

2.5 Research perspectives

We conclude this chapter by mentioning possible developments of this topic, some of which are the object of works in preparation.

2.5.1 Geometry of compact quotients of $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$

As we have extensively discussed, we have reached a fairly good understanding of the geometry of compact quotients of rank 1 group spaces. We now want to focus on the other family of reductive homogeneous spaces which are known to admit non-standard quotients, namely $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$. In a work in preparation with Monclair and Schlenker [149], we investigate further the relation between compact quotients of $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ and convex-GHC anti-de Sitter manifolds. We construct exotic compact quotients and prove the geometric fibration conjecture for those quotients.

Recall that Barbot proved that continuous deformations of convex-GHC subgroups of $\mathrm{SO}(k,2)$ remain convex-GHC, and conjectured that every convex-GHC subgroup of $\mathrm{SO}(k,2)$ is virtually a deformation of a uniform lattice in $\mathrm{SO}(k,1)$. Our main resultin [149] will be the construction, for any $\mathfrak{k} \geq 4$, of convex-GHC subgroups $\Gamma \subset \mathrm{SO}(k,2)$ isomorphic to the fundamental group of a *Gromov–Thurston manifold* [79].

These manifolds are among the few examples of closed negatively curved manifolds (in any dimension greater than 3) which are not homeomorphic to quotients of a symmetric space. In [149], we prove that their are isomorphic to convex-GHC groups. In particular, these convex-GHC groups are not deformations of lattices in $\mathrm{SO}(k,1)$. They thus disprove Barbot's conjecture in every dimension $k \geq 4$. Other counter-examples have been constructed by Lee and Marquis for $4 \leq k \leq 8$ using hyperbolic Coxeter groups [128].

Remark 2.5.1. Barbot's conjecture is known to hold for $k = 2$ by the work of Mess. For $k = 3$ it is still open and must be more subtle than in higher dimension since, by Perelman's hyperbolization theorem, every convex-GHC subgroup of $\mathrm{SO}(3,2)$ is isomorphic to a lattice in $\mathrm{SO}(3,1)$.

Since, by Theorem 2.3.13, convex-GHC subgroups of $\mathrm{SO}(2d,2)$ act properly discontinuously and cocompactly on $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$, these counter-examples to Barbot's conjecture give the following:

Theorem 2.5.2 (Monclair–Schlenker–Tholozan [149]). *For every* $d \geq 2$*, the reductive homogeneous space* $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ *admits compact quotients by Zariski dense subgroups of* $\mathrm{SO}(2d,2)$ *that are not isomorphic to any lattice.*

We will also give a more concrete understanding of the relation between a convex-GHC anti-de Sitter manifold of dimension $2d+1$ and the corresponding compact quotient of $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$. We first interpretate $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ as the space of *unit timelike Killing fields* on AdS^{2d+1} (i.e. vector fields X infinitesimally preserving the Lorentzian metric g and satisfying $g(X,X) = -1$ everywhere). Now, given a strictly convex spacelike hypersurface $\mathcal{H} \subset \mathrm{AdS}^{2d,1}$, we prove that any unit timelike Killing field is orthogonal to $\mathcal{H}$ at exactly one point. This defines a projection map $\pi_{\mathcal{H}}$ from $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ to $\mathcal{H}$ which turns out to be a geometric fibration.

Assume moreover that some $\Gamma \subset \mathrm{SO}(2d,2)$ acts properly discontinuously and cocompactly $\mathcal{H}$. Then this projection is Γ-equivariant. This argument gives a new explanation of the fact that Γ acts properly discontinuously and cocompactly on $\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ and proves the geometric fibration conjecture for such quotients.

Theorem 2.5.3 (Monclair–Schlenker–Tholozan [149]). *Let* Γ *be a subgroup of* $\mathrm{SO}(2d,2)$ *acting properly discontinuously and cocompactly on a convex spacelike hypersurface* $\mathcal{H}$*. Then* $\Gamma\backslash\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ *admits a geometric fibration over* $\Gamma\backslash\mathcal{H}$*.*

Finally, one could apply Theorem 2.4.12 to interpretate the volume of $\Gamma\backslash\mathrm{SO}(2d,2)/\mathrm{U}(d,1)$ as a characteristic class. We obtain the following result, that we hope to include in some future work:

Theorem 2.5.4. *Let* Γ *be a subgroup of* $\mathrm{SO}(2d,2)$ *acting properly discontinuously and cocompactly on a convex spacelike hypersurface* $\mathcal{H}$*of* AdS^{2d+1}*. Then*

$$\mathbf{Vol}(\Gamma\backslash\mathrm{SO}(2d,2)/\mathrm{U}(d,1) = C_d \left|\chi(\Gamma\backslash\mathcal{H})\right| \ ,$$

where χ *denotes the Euler characteristic and* C_d *is a constant depending only on the normalization of the volume form.*

Volume and Chern–Simons theory

Shortly after hearing of my volume computation for closed AdS 3-manifolds (Theorem 2.4.11), Labourie gave it an alternative proof using Chern–Simons theory of secondary characteristic classes. In some future work, we intend to detail his argument, which extends to a broader context and proves the rigidity of the volume in full generality.

Theorem 2.5.5. *Let* G/H *be a reductive homogeneous space and* Γ *a discrete subgroup of* G *acting properly discontinuously and cocompactly on* G/H*. Then the volume of* $\Gamma\backslash G/H$ *is constant under deformations of* Γ*.*

Let us sketch a proof of this theorem, which we hope to detail in a forthcoming work.

Proof. First, one can reduce to the case of group spaces by chosing a uniform lattice Λ in H and replacing $\Gamma\backslash G/H$ by $\Gamma\backslash G/\Lambda$, whose volume satisfies

$$\mathbf{Vol}(\Gamma\backslash G/\Lambda) = \mathbf{Vol}(\Gamma\backslash G/H) \cdot \mathbf{Vol}(H/\Lambda)$$

(for suitable scalings of the volume forms). Now the tangent space to a Lie group G admits two bi-invariant connections ∇_L and ∇_R, given respectively by left and right parallelism. Cartan discribed explicitly the algebra of bi-invariant forms on G. This algebra is generated by odd-dimensional forms that can be written as "Chern–Simons forms", i.e.

$$\int_{t=0}^{1} P(\dot{\nabla}_t, \Omega_{\nabla_t}, \ldots, \Omega_{\nabla_t})$$

where P is a G-invariant polynomial on $\mathfrak{g}$, ∇_t is the connection $(1-t)\nabla_L + t\nabla_R$ and Ω_{∇_t} denotes its curvature tensor.

These forms factor to compact quotients of G, and one can prove that their cohomology class is invariant under deformation of the flat connections ∇_L and ∇_R within the space of flat connections. Since the volume form is a polynomial combination of those forms, one deduces its rigidity. □

This argument can be pushed further in various directions. Theorem 2.5.5 can for instance be extended to manifolds locally modelled on G/H.[2] In a slightly different direction one can prove the following:

Theorem 2.5.6. *Let Γ be a finitely generated group, G a semisimple Lie group and α a G-invariant differential form on its symmetric space X (which can be seen as a continuous cohomology class of the group G). Then the map*

$$\rho \mapsto [\rho^* \alpha]$$

from $\mathrm{Hom}(\Gamma, G)$ *to* $\mathrm{H}^\bullet(\Gamma, \mathbb{R})$ *is locally constant.*

This statement does not seem to have been established before with this degree of generality. As mentioned in Section 1.2.3, it is known when α is a Chern–Weil form. The general statement will follow from interpreting other G-invariant forms with Chern–Simmons theory.

2. Of course, this extension is potentially empty if the reductive Markus conjecture is true.

2.5.2 Quotients of finite volume

We decided to focus here on compact quotients for a more coherent approach, but one might be interested in removing the compactness assumption. In particular, it is natural to ask which results extend to quotients of finite volume.

Without entering into the details, let us mention that, while the cohomological obstructions to the existence of compact quotients rely on compactness in an essential way, some of the geometric obstructions (such as Benoist's) have a larger scope, and the dynamical obstructions are likely to generalize to the finite volume case.

Concerning the construction of non-standard and exotic compact quotients for $\mathrm{SO}(d,1)$, $\mathrm{SU}(d,1)$ and $\mathrm{SO}(2k,2)/\mathrm{U}(k,1)$, it seems that all of them extend to construct non-standard and non-compact quotients of finite volume.

On the other side, very little is known toward a geometric characterization of quotients of finite volume. In particular, the following crucial question remains unanswered:

Question 2.5.7. *Let Γ be a discrete subgroup of G acting properly discontinuously on G/H with finite covolume. Is Γ finitely generated ?*

When G/H is Riemannian, the answer to this question is notoriously positive. If G has rank 1, it follows from a geometric description of the ends of $\Gamma\backslash G/H$ (see [163]). When G has higher rank, it follows for instance from Kazhdan's property T (see [16]), or can be proved more geometrically using the existence of a coarse fundamental domain. (However, the latter approach relies on Margulis's arithmeticity theorem.)

The higher rank tools (arithmeticity or Kazhdan's property T) seem useless in the pseudo-Riemannian setting, where the non-standard examples developed so far tend to behave like rank one lattices. Question 2.5.7 is still open, which shows how little is known in general about those finite volume quotients.

A bit more can be said about quotients of the group spaces $\mathrm{SO}_0(d,1)$, for which a lot is known thanks to the work of Guéritaud–Kassel. In [81], they study more generally quotients of the form $\Gamma_\rho\backslash\mathrm{SO}_0(d,1)$, where Γ is a *geometrically finite* subgroup of $\mathrm{SO}_0(d,1)$ (in the sense of [28]), and ρ a representation of Γ into $\mathrm{SO}_0(d,1)$. Their admissibility criterion generalizes to this context, namely, Γ_ρ acts properly discontinuously on $\mathrm{SO}_0(d,1)$ if and only if ρ is contracting. We call such quotients *geometrically finite*.

Geometrically finite quotients admit a geometric fibration over $\Gamma\backslash\mathbb{H}^d$ and our computation of the volume works in this situation.

Theorem 2.5.8 (Tholozan, [191])**.** *We have*

$$\mathbf{Vol}(\Gamma_\rho\backslash\mathrm{SO}_0(d,1)) = A_d \int_{\Gamma\backslash\mathbb{H}^d} vol_{\mathbb{H}^d} + (-1)^d f^* vol_{\mathbb{H}^d} \ ,$$

where $f : \mathbb{H}^d \to \mathbb{H}^d$ *is any* ρ*-equivariant Lipschitz map.*

If we choose f to be contracting, we obtain in particular that

$$\mathbf{Vol}(\Gamma_\rho \backslash \mathrm{SO}_0(d,1)) \geq A_d(1-\lambda)\mathbf{Vol}(\Gamma\backslash\mathbb{H}^d)$$

for some $\lambda < 1$. Therefore, this volume is finite if and only if Γ is a lattice in $\mathbb{H}^d$.

Using known volume rigidity results for representations of hyperbolic lattices [34, 103], we deduce the volume rigidity of geometrically finite quotients of $\mathrm{SO}_0(d,1)$ for $d \geq 3$. In contrast, for $d = 2$, one obtains that the volume of the quotients $\Gamma_\rho\backslash\mathrm{SO}_0(2,1)$ can vary continuously in the interval $\big(0, 2\mathbf{Vol}(\Gamma\backslash\mathbb{H}^2)\big)$ when ρ varies in the set of admissible representations.

The main obstacle to a complete description of all finite volume quotients of $\mathrm{SO}_0(d,1)$ is Guéritaud–Kassel's geometric finiteness assumption. Hence the following question:

Question 2.5.9. *Let* Γ *be a discrete subgroup of* $\mathrm{SO}_0(d,1)$ *and* $\rho : \Gamma \to \mathrm{SO}_0(d,1)$ *an admissible representation such that* $\Gamma_\rho\backslash\mathrm{SO}_0(d,1)$ *has finite volume. Is* Γ *geometrically finite?*

When $d = 2$, Γ is geometrically finite if and only if it is finitely generated, and this question is equivalent to Question 2.5.7. To answer it, one can try to generalize Guéritaud–Kassel's results to quotients $\Gamma_\rho\backslash\mathrm{SO}_0(d,1)$ which are not geometrically finite. Guéritaud–Kassel's admissiblity criterion does not hold here. Instead, one can show the following weaker statements:

Proposition 2.5.10. *If* ρ *is admissible, then there exists* $f : \mathbb{H}^d \to \mathbb{H}^d$ *which is* ρ*-equivariant and weakly contracting, i.e.*

$$d(f(x), f(y)) < d(x,y)$$

for all $x \neq y$*.*

Conversely, we have

Proposition 2.5.11. *If there exists a* ρ*-equivariant weakly contracting map* $f : \mathbb{H}^d \to \mathbb{H}^d$*, then* Γ_ρ *acts properly discontinuously on the domain*

$$U_f = \{g \in \mathrm{SO}_0(d,1) \ \textit{s.t.}\ g\circ f \ \textit{has a fixed point}\}\ .$$

Moreover, $\Gamma_\rho\backslash U_f$ *admits a geometric fibration over* $\Gamma\backslash\mathbb{H}^d$*.*

We strongly believe that Γ_ρ acts properly discontinuously on $\mathrm{SO}_0(d,1)$ if and only if f can be chosen so that $U_f = \mathrm{SO}_0(d,1)$.

Conjecture 2.5.12. *The group* Γ_ρ *acts properly discontinously on* $\mathrm{SO}_0(d,1)$ *if and only if there exists a* ρ*-equivariant and weakly contracting map* $f : \mathbb{H}^d \to \mathbb{H}^d$ *such that* $U_f = \mathrm{SO}_0(d,1)$*.*

One can show that $U_f = \mathrm{SO}_0(d,1)$ if and only if for every $C > 0$ there exists $R > 0$ such that $f(B(o,R)) \subset B(o, R-C)$. It thus requires f to be "contracting enough" at infinity (typically, $\lambda(r)$-Lipschitz on $B(o,r)$ for some function $\lambda < 1$ such that $\int_{\mathbb{R}_+} 1 - \lambda(r) = +\infty$). It is thus more constraining than being weakly contracting, but weaker than being strongly contracting.

Now, if f is weakly contracting, our volume computation in [191] works and gives:

$$\mathbf{Vol}(\Gamma_\rho \backslash U_f) = A_d \int_{\Gamma\backslash\mathbb{H}^d} \left(vol_{\mathbb{H}^d} + (-1)^d f^* vol_{\mathbb{H}^d} \right) \ .$$

If f is sufficiently close to being isometric on complements of compact sets of $\Gamma\backslash\mathbb{H}^d$, then it could happen (and does indeed in some cases) that the two terms of the intergrand cancel out asymptotically so that the integral is finite. Thus Question 2.5.9 boils down to asking whether f can be both:

— "contracting enough" asymptotically so that $U_f = \mathrm{SO}_0(d,1)$,

— "isometric enough" asymptotically so that $\Gamma\backslash U_f$ has finite volume.

Unfortunately, even for $d = 2$, where things are easy to manipulate, Fanny Kassel and I haven't been able to answer positively of negatively. In all the examples we tried to study explicitly the two conditions seemed incompatible, but the extreme flexibility of hyperbolic surfaces of infinite type makes it hard to convert this into a general argument.

Chapter 3

Surface group representations

In this chapter, we investigate the relations between the topology and geometry of surfaces and the linear representations of their fundamental group. There are several motivations to this study:

— A first motivation is historical: surface group representations predate the invention of the fundamental group. They have been studied since XIX$^{\text{th}}$ century because they appear as monodromies of differential equations of a complex variable.

— Fundamental groups of surfaces of finite type are lattices in $\mathrm{PSL}(2, \mathbb{R})$. Contrary to higher rank lattices (and to some extent, to other lattices in rank 1 Lie groups), they are highly flexible. Understanding their linear representations thus constitutes an entire part of the study of linear representations of lattices.

— Through the theory of (G, X)-structures, representations of surface groups play an important role in low dimensional geometry. Representations into $\mathrm{PSL}(2, \mathbb{C})$, for instance, are crucial in Thurston's hyperbolization of Haken 3-manifolds.

— Surface groups are ubiquitous: Hamenstädt [86] and Kahn–Labourie–Mozes [93], generalizing the work of Kahn–Marcovic [94], proved that many higher rank lattices contain a lot of (closed) surface groups.

— Surface groups are the first examples of Kähler groups. The *non-Abelian Hodge correspondence* establishes an analytic bijection between their character varieties and the moduli spaces of Higgs bundles over Riemann surfaces [178]. These moduli spaces carry a rich and interesting complex geometry. In particular, they form some of the few examples of *algebraically completely integrable systems* [88].

— Surface groups have a large outer automorphism group (the *mapping class group* of the surface) which acts on character varieties preserving their symplectic structure. These are rich and interesting dynamical systems, which are not well-understood yet.

After a few general considerations on surfaces and the representations of their fundamental groups, I will present certain character varieties of surface groups, particularly the so-called *higher Teichmüller spaces* (Section 3.1). I will then recall briefly the relation between harmonic maps, minimal embeddings and Higgs bundles over a Riemann surfaces, and explain how I used these as tools to understand fine properties of certain classes of surface group representations (Section 3.2). In Section 3.3, I will present my recent works with Bertrand Deroin and Jérémy Toulisse which constructs bounded relative character varieties of punctured spheres. Finally, Section 3.4 is devoted to my ongoing projects related to surface group representations: the first one (Section 3.4.1) presents different points of view on what I call a *highest Teichmüller space*, i.e. and infinite dimensional space in which all higher Teichmüller spaces embed. The second project (Section 3.4.2) discusses an approach to understanding holonomies of hyperbolic metrics with cone singularities that we are developing with Bertrand Deroin. The third project (Section 3.4.3) inscribes my recent work on bounded components of relative character varieties into the more general study of bounded mapping class group orbits.

3.1 Surface groups and their representations

3.1.1 Surfaces and their Teichmüller spaces

Throughout this chapter, we will denote by Σ_g a closed oriented surface of genus g, and by $\Sigma_{g,n}$ the *n-punctured surface of genus g*, i.e. the surface Σ_g with n points removed. We denote by Γ_g the fundamental group of Σ_g and by $\Gamma_{g,n}$ the fundamental group of $\Sigma_{g,n}$. We will only consider surfaces with negative Euler characteristic, i.e.

$$\chi(\Sigma_{g,n}) = 2 - 2g - n < 0 \ .$$

(This only excludes the sphere, the open disc, the cylinder and the closed torus.)

We will denote by $[\Gamma_{g,n}]$ the set of conjugacy classes in $\Gamma_{g,n} \backslash \{\mathbf{1}\}$. This set is in bijection with the set of free homotopy classes of closed curves on $\Sigma_{g,n}$. A non-trivial closed curve on $\Sigma_{g,n}$ (or the corresponding class in $[\Gamma_{g,n}]$) is called *simple* if it is (or is freely homotopic to) an embedding of the circle. Simple closed curves allow to cut surfaces into subsurfaces. More precisely, the complement of a simple closed curve in $\Sigma_{g,n}$ is homeomorphic either to $\Sigma_{g-1,n+2}$ or to the disjoint union $\Sigma_{g_1,n_1} \sqcup \Sigma_{g_2,n_2}$ with $g_1 + g_2 = g$ and $n_1 + n_2 = n + 2$.

When $n > 0$, the fundamental group $\Gamma_{g,n}$ is a free group in $2g + n - 1$ generators. In particular, two punctured surfaces with the same Euler characteristic (such as $\Sigma_{0,3}$ and $\Sigma_{1,1}$) have the same fundamental group. To distinguish between those groups, one needs to single out the *peripheral classes* $c_1, \dots, c_n$ of $\Gamma_{g,n}$, i.e. the conjugacy classes corresponding to the curves circling each puncture counter-clockwise.

The *(pure) mapping class group* of the surface $\Sigma_{g,n}$ is the group of connected components of the group of homeomorphisms of Σ_g fixing the punctures (or equivalently, homeomorphisms of $\Sigma_{g,n}$ fixing its ends). We denote it by $\mathrm{MCG}_{g,n}$. The mapping class group embeds in the outer automorphism group

$$\mathrm{Out}(\Gamma_{g,n}) = \mathrm{Aut}(\Gamma_{g,n})/\mathrm{Inn}(\Gamma_{g,n}) \ .$$

When $n = 0$, this embedding is an isomorphism, by the Dehn–Nielsen–Baer theorem. For surfaces with punctures, this embedding identifies $\mathrm{MCG}_{g,n}$ with the subgroup of $\mathrm{Out}(\Gamma_{g,n})$ fixing the peripheral classes (see [61]).

Teichmüller space

As a convention, a *complex structure* on $\Sigma_{g,n}$ will be a complex structure that extends to Σ_g and is compatible with the orientation of $\Sigma_{g,n}$. The *Teichmüller space* of the surface $\Sigma_{g,n}$, denoted $\mathcal{T}_{g,n}$, is the space of complex structures on Σ_g modulo isotopies *fixing the punctures.* Teichmüller proved that it is a complex manifold of complex dimension $3g - 3 + n$. The group $\mathrm{MCG}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by pushing forward complex structures. This action is properly discontinuous and the quotient $\mathcal{M}_{g,n} = \mathrm{MCG}_{g,n} \backslash \mathcal{T}_{g,n}$ is the *moduli space* of Riemann surfaces of genus g with n punctures.

Recall that a complex structure on $\Sigma_{g,n}$ defines a *conformal structure* on Σ_g, i.e. a conformal class on Riemannian metrics – namely those of the form $e^\sigma \mathrm{d}z\mathrm{d}\bar{z}$ in a local complex coordinate z. Conversely, the isothermal coordinate theorem of Gauss asserts that every Riemannian metric on a surface is locally conformal to a flat metric, and thus defines (together with the orientation) a unique complex structure.

Now, the uniformization theorem of Riemann surfaces (due to Klein and Poincaré for closed surfaces and to Poincaré and Koebe in general [54]) asserts that, given any complex structure J on $\Sigma_{g,n}$, there exists a unique complete conformal metric on $\Sigma_{g,n}$ which is *hyperbolic*, i.e. of curvature -1. In other words, the Riemann surface $(\Sigma_{g,n}, J)$ is biholomorphic to $j(\Gamma_{g,n})\backslash\mathbb{H}^2$, where $j : \Gamma_{g,n} \to \mathrm{Isom}_+(\mathbb{H}^2)$ is discrete and faithful. Moreover, since the complex structure on $\Sigma_{g,n}$ extends to Σ_g, the uniformizing hyperbolic metric has finite volume, so that the image of j is a lattice in $\mathrm{Isom}_+(\mathbb{H}^2)$. The representation j is called *Fuchsian.*

The uniformization theorem gives two alternative points of view on $\mathcal{T}_{g,n}$:

— $\mathcal{T}_{g,n}$ is the space of complete hyperbolic metrics of finite volume on $\Sigma_{g,n}$ modulo isotopy.

— $\mathcal{T}_{g,n}$ is the space of Fuchsian representations of $\Gamma_{g,n}$ modulo conjugation.

The geometry of $\mathcal{T}_{g,n}$ is rich of this diversity of points of view. It carries a complex structure, first constructed by Teichmüller, which is best understood via the theory of quasi-comformal maps between Riemann surfaces. Teichmüller also introduced a Finsler metric on $\mathcal{T}_{g,n}$ which turns out to be its *Kobayashi metric*:

Definition 3.1.1. Let $[g_1]$ and $[g_2]$ be two points on $\mathcal{T}_g$, seen as conformal structures on $\Sigma_{g,n}$. The Teichmüller distance between $[g_1]$ and $[g_2]$ is given by

$$d_{\mathcal{T}}([g_1],[g_2]) = \log\left(\inf_{g_2'}\left(\inf\{K \mid \frac{1}{K}g_1 \leq g_2' \leq K g_1\}\right)\right) ,$$

where the first infimum is taken over all metrics g_2' that are isotopic to a metric in the conformal class $[g_2]$.

It also carries a natural Riemannian metric, called the *Weil–Petersson metric*, which is the quotient modulo isotopy of the $\mathcal{L}^2$ metric on the space of metrics of constant curvature -1. We denote it by h_{WP}. The Weil–Petersson metric is Kähler [4], has negative sectional curvature [5, 206] and the associated symplectic structure coincides with the Atiyah–Bott symplectic structure on the space of Fuchsian representations (see Section 3.1.2).

Finally, a third metric structure on $\mathcal{T}_g$ is given by *Thurston's asymmetric metric* [196]. Let j_1 and j_2 be two points in $\mathcal{T}_g$, seen as Fuchsian representations of Γ_g.

Definition 3.1.2. Thurston's asymmetric distance between j_1 and j_2 is the logarithm of the infimum of Lipschitz constants of (j_1, j_2)-equivariant maps from $\mathbb{H}^2$ to $\mathbb{H}^2$. Equivalently,

$$d_{Th}(j_1, j_2) = \log\left(\sup_{\gamma\in[\Gamma]} \frac{L_{j_2}(\gamma)}{L_{j_1}(\gamma)}\right) .$$

Geometry and dynamics of Fuchsian representations

Fuchsian representations, as holonomies of hyperbolic metrics on $\Sigma_{g,n}$, are in some sense the "most geometric" representations of a surface group. When studying other linear representations of $\Gamma_{g,n}$, a constant preocupation will be to compare their geometric and dynamical properties to those of Fuchsian representations. Let us mention some of those properties.

Recall that we denote by $L_j : [\Gamma_{g,n}] \to \mathbb{R}_+$ the length spectrum of a Fuchsian representation $j : \Gamma_{g,n} \to \mathrm{Isom}_+(\mathbb{H}^2)$, i.e.

$$L_j(\gamma) = \inf_{x \in \mathbb{H}^2} d(x, j(\gamma) \cdot x) \ .$$

It coincides with the *highest weight length spectrum* of j (see Section 1.2.4) when we identify $\mathrm{Isom}_+(\mathbb{H}^2)$ with $\mathrm{SO}_0(2,1) \subset \mathrm{SL}(3, \mathbb{R})$. It also coincides with the length spectrum of the hyperbolic metric g_{hyp} on $\Sigma_{g,n}$ corresponding to j, in the sense that $L_j(\gamma)$ is the length of the unique closed geodesic in the free homotopy class corresponding to γ.

Several geometric properties of hyperbolic surfaces can be expressed in terms of their length spectrum. Let us mention three such properties, namely a universal bound for the systole, the existence of a Bers constant and the collar lemma.

Proposition 3.1.3 (Systole)**.** *There exists a constant $\sigma_{g,n}$ such that, for any Fuchsian representation $j : \Gamma_{g,n} \to \mathrm{Isom}_+(\mathbb{H}^2)$, there exists a non-peripheral closed curve γ with*

$$L_j(\gamma) \leq \sigma_{g,n} \ .$$

The optimal constant $\sigma_{g,n}$ is the largest systole in $\mathcal{T}_{g,n}$.

A *pair of pants decomposition* of $\Sigma_{g,n}$ is a collection of $3g-3+n$ disjoint simple closed curves on $\Sigma_{g,n}$ whose complement is a union of $2g-2+n$ disjoint thrice punctured spheres.

Proposition 3.1.4. *There exists a constant $B_{g,n}$ such that, for any Fuchsian representation $j : \Gamma_{g,n} \to \mathrm{Isom}_+(\mathbb{H}^2)$, there exists a pair of pants decomposition $(\gamma_i)_{1 \leq i \leq 3g-3+n}$ of $\Sigma_{g,n}$ such that*

$$L_j(\gamma_i) \leq B_{g,n}$$

for all i. The optimal constant $B_{g,n}$ is the Bers constant of $\Sigma_{g,n}$.

We say that two closed curves γ and η on $\Sigma_{g,n}$ intersect essentially if they are not freely homotopic to disjoint curves. Keen's collar lemma quantifies the fact that two intersecting geodesics on a hyperbolic surface cannot both be short.

Lemma 3.1.5 (Keen's Collar lemma)**.** *Let γ and η be two closed curves on $\Sigma_{g,n}$ that essentially intersect. Then, for every Fuchsian representation $j : \Gamma_{g,n} \to \mathrm{Isom}_+(H^2)$, we have*

$$\sinh\left(\frac{L_j(\gamma)}{2}\right) \cdot \sinh\left(\frac{L_j(\eta)}{2}\right) \geq 1 \ .$$

Finally, let us recall that the entropy $\mathscr{H}(j)$ of a Fuchsian representation is equal to 1.

In Section 3.2, we will present several comparison results which imply some generalizations of the above properties to certain surface group representations into other Lie groups.

3.1.2 Character varieties and their symplectic geometry

We introduced character varieties of discrete groups into semisimple Lie groups in Section 1.2.2. Let us now discuss the specific properties of character varieties of surface groups. We start by focusing on fundamental groups of closed surfaces. More details can be found in [122].

For convenience, let us assume that our semisimple group G is linear algebraic. Recall that the space $\mathfrak{X}(\Gamma_g, G)$ is the largest Hausdorff quotient of $\mathrm{Hom}(\Gamma_g, G)$ under the conjugation action of G. It admits a structure of real semi-algebraic set. We call a representation $\rho : \Gamma_g \to G$ *strongly irreducible* if its centralizer is reduced to the center of G.

Proposition 3.1.6. *The character variety $\mathfrak{X}(\Gamma_g, G)$ has dimension $(2g - 2)\dim(G)$. If $\rho : \Gamma_g \to G$ is strongly irreducible, then $\mathfrak{X}(\Gamma_g, G)$ is smooth at $[\rho]$.*

One of the nicest features of character varieties of surface groups is that they carry a natural symplectic structure (on the smooth locus), introduced first by Atiyah–Bott [9] and further studied by Goldman [73, 75].

Recall that the (algebraic) tangent space to $\mathfrak{X}(\Gamma_g, G)$ at $[\rho]$ is canonically identified with the twisted cohomology group $\mathrm{H}^1(\Gamma_g, \mathrm{Ad}_\rho)$. The Killing form on $\mathfrak{g}$ induces an antisymmetric bilinear map

$$\mathrm{H}^1(\Gamma_g, \mathrm{Ad}_\rho) \times \mathrm{H}^1(\Gamma_g, \mathrm{Ad}_\rho) \to \mathrm{H}^2(\Gamma_g, \mathbb{R}) \simeq \mathrm{H}^2(\Sigma_g, \mathbb{R}) ,$$

and the integration over Σ_g makes it a bilinear form on $\mathrm{H}^1(\Gamma_g, \mathrm{Ad}_\rho)$. This gives a 2-form ω_{AB} on (the smooth locus of) $\mathfrak{X}(\Gamma_g, G)$.

In [9], Atiyah and Bott interpret $(\mathfrak{X}(\Gamma_g, G), \omega_{AB})$ as the symplectic reduction of the space of all connections on a principal G-bundle (which is an inifinite dimensional affine space with a natural translation invariant symplectic form) under the action of the group of Gauge transformations. This shows in particular that ω_{AB} is closed and non-degenerate at smooth points.

Goldman studied in details the symplectic geometry of $\mathfrak{X}(\Gamma_g, G)$. In particular, he found an explicit description of the Hamiltonian flow associated to the functions:

$$\begin{array}{rccc} F_{\gamma,P} : & \mathfrak{X}(\Gamma_g, G) & \to & \mathbb{R} \\ & [\rho] & \mapsto & P(\rho(\gamma)) \end{array}$$

where $P : G \to \mathbb{R}$ is a smooth function invariant under conjugation and γ is a simple closed curve [75]. He also gave formulae for the Poisson bracket of $F_{\gamma,P}$ and $F_{\eta,P}$, building a bridge with Topological Quantum Field Theory.

Note finally that the action of $\mathrm{Aut}(\Gamma_g)$ on $\mathrm{Hom}(\Gamma_g, G)$ by precomposition of representations factors to an analytic action of MCG_g on $\mathfrak{X}(\Gamma_g, G)$ which preserve the Atiyah–Bott symplectic form.

Relative character varieties of punctured surfaces

The symplectic structure of $\mathfrak{X}(\Gamma_g, G)$ does not extend to punctured surface groups in a straightforward way, for instance because $\mathrm{H}^2(\Sigma_{g,n}, \mathbb{R}) = \{0\}$ for $n > 1$. It does extend, however, to the *relative character varieties* that we introduce here.

Let $\widehat{C}(G)$ denote the set of conjugacy classes in G and $C(G)$ its largest Hausdorff quotient. (In other words, $C(G)$ is th character variety of the group $\mathbb{Z}$.) Recall that we denote by $c_1, \ldots, c_n$ the peripheral curves in $\Sigma_{g,n}$ (as well as the associated conjugacy classes in $\Gamma_{g,n}$).

There is a semi-algebraic *restriction map*

$$\begin{array}{rrcl} \mathrm{Res}: & \mathfrak{X}(\Gamma_{g,n}, G) & \to & C(G)^n \\ & \rho & \mapsto & ([\rho(c_i)])_{1 \leq i \leq n} \,. \end{array}$$

The fibers of this map are called the *relative character varieties* of $\Gamma_{g,n}$ into G. Given $\mathbf{a} = (a_1, \ldots, a_n) \in C(G)$, we will denote by $\mathfrak{X}_{\mathbf{a}}(\Gamma_{g,n}, G)$ the preimage of $\mathbf{a}$ by Res.

For a generic $\mathbf{a}$, each a_i consists of a single conjugacy class in G of dimension $\dim G - \mathbf{rk}_{\mathbb{C}}(G)$. In general, a_i is stratified by finitely many conjugacy classes, and this stratification induces a stratification of $\mathfrak{X}_{\mathbf{a}}(\Gamma_{g,n}, G)$. Given $\widehat{a}_i \in a_i$ for all i, we denote by $\mathfrak{X}_{\widehat{\mathbf{a}}}(\Gamma_{g,n}, G) \subset \mathfrak{X}_{\mathbf{a}}(\Gamma_{g,n}, G)$ the conjugacy classes of reductive representations $\rho : \Gamma_{g,n} \to G$ such that $\rho(c_i)$ is conjugate to $\widehat{a}_i$ for all i, and call it a *relative stratum*.

Proposition 3.1.7. *Let $\rho : \Gamma_{g,n} \to G$ be a strongly irreducible representation. Then $[\rho]$ is a smooth point of its relative stratum.*

Finally, the Atiyah–Bott symplectic form can now be defined on (the smooth locus of) relative strata (see [77]).

To summarize, the whole character variety $\mathfrak{X}(\Gamma_{g,n}, G)$ admits a (singular) foliation whose leaves carry a symplectic structure. This is essentially the same as a *Poisson structure* (i.e. a Poisson bracket on the space of regular functions). While the whole group $\mathrm{Out}(\Gamma_{g,n})$ acts on $\mathfrak{X}(\Gamma_{g,n}, G)$, the subgroup preserving the relative strata is exactly the pure mapping class group $\mathrm{MCG}_{g,n}$, which also preserves the Poisson structure (hence the symplectic structure of each stratum).

We now focus on closed surface groups and present in more details the character varieties that will interest us. We will come back to relative character varieties in Section 3.4.3.

3.1.3 The $\mathrm{PSL}(2, \mathbb{R})$ character variety

A character variety of particular historical importance is the $\mathrm{PSL}(2, \mathbb{R})$-character variety $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$. It already played a key role in Poincaré and Klein's continuity method to uniformize compact Riemann surfaces, but it is only in the eighties that its topology was completely described by the work of Goldman [76] and Hitchin [89].

Recall that the Uniformization theorem of Klein–Poincaré identifies the unoriented Teichmüller space $\mathcal{T}_g^{\pm}$ with the subset $\mathfrak{X}_{Fuchs}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ of $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ consisting of conjugacy classes of Fuchsian representations. It was already known by Klein and Poincaré that $\mathfrak{X}_{Fuchs}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ forms two connected components of $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ which are permuted by the outer automorphism of $\mathrm{PSL}(2,\mathbb{R})$ (i.e. the conjugation by an orientation-reversing isometry of $\mathbb{H}^2$). Since none of them contains the trivial representation, this shows that $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ has several connected components (even modulo the orientation switch).

It turns out that these connected components are distinguished by their Euler class, which can be seen as a locally constant map $\mathbf{eu}$ from $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ to $\mathbb{Z}$. Milnor proved that it takes exactly all the integral values in the range of the Euler characteristic of the surface:

Theorem 3.1.8 (Milnor–Wood inequality [147, 208])**.** *Every $\rho \in \mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ satisfies*

$$|\mathbf{eu}(\rho)| \leq 2g-2 \ .$$

Conversely, for every integer $2-2g \leq k \leq 2g-2$, there exists $\rho \in \mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ such that $\mathbf{eu}(\rho) = k$.

Fuchsian representations have extremal Euler class (i.e. $\pm(2g-2)$). In the 70s, Thurston gave an argument to prove the converse, building on Gromov's notion of simplicial volume. This was also proved in a more elementary way by Goldman in his thesis [72]. Finally, in the late 80's, Goldman showed that the preimages of the Euler class are connected while, at about the same time, Hitchin described their topology, as the first spectacular application of the non-abelian Hodge correspondance (see Section 3.2.1). Let $\mathfrak{X}_k(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ denote the set of classes of representations of Euler class k.

Theorem 3.1.9.

- *For every $2-2g \leq k \leq 2g-2$, the space $\mathfrak{X}_k(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ is connected. (Goldman [76])*
- *For $k \geq 1$, the space $\mathfrak{X}_k(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ is homeomorphic to a complex vector bundle of rank $3g-3-k$ over the k^{th} symmetric power of Σ_g. (Hitchin [89])*

Remark 3.1.10. Applying the parabolic non-Abelian Hodge correspondance, Mondello recently generalized Hitchin's description of the $\mathrm{PSL}(2,\mathbb{R})$ character variety to relative character varieties of punctured surface groups.

Let us now discuss the main open problems concerning $\mathrm{PSL}(2,\mathbb{R})$-character varieties.

Simple closed curves and mapping class group dynamics

We already saw that the mapping class group MCG_g acts properly discontinuously on $\mathcal{T}_\pm(\Sigma_g) \simeq \mathfrak{X}_{Fuchs}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$. The quotient space is the moduli space of Riemann surfaces of genus g. In contrast, the mapping class group action on non-extremal components of the character variety is very chaotic. In fact, it is believed to be ergodic with respect to the measure ω_{AB}^{3g-3}.

Conjecture 3.1.11 (Goldman)**.** *Let U be a MCG_g-invariant subset of $\mathfrak{X}_k(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$, with $|k| < 2g-2$ (and $g \geq 3$). Then either U or its complement has measure* 0.

The exception $g = 2$ was raised by Marché and Wolf [136] who noticed that the subset of $\mathfrak{X}_0(\Gamma_2, \mathrm{PSL}(2,\mathbb{R}))$ formed by irreducible representations has two MCG_2-invariant connected components on which the action is ergodic. They also proved the ergodicity of MCG_2 on $\mathfrak{X}_1(\Gamma_2, \mathrm{PSL}(2,\mathbb{R}))$, thus giving a complete answer to Goldman's conjecture in genus 2.

Goldman's conjecture is deeply related to another conjecture of Bowditch:

Conjecture 3.1.12 (Bowditch)**.** *Let $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ be a non-Fuchsian representation. Then there exists a simple closed curve $\gamma \in [\Gamma_g]$ such that $\rho(\gamma)$ is not hyperbolic.*

In fact, Marché and Wolff, following an argument of Goldman, proved that MCG_g acts ergodically on connected components of the open MCG_g-invariant domain of representations that map some simple closed curve to an elliptic element. This shows in particular that Goldman's conjecture is in fact equivalent to the slightly weaker "generic" Bowditch conjecture.

Multiformization

A *branched hyperbolic structure* on a surface Σ is a path metric which is locally isometric to a hyperbolic cone with angle a multiple of 2π. To be more precise, let g_{hyp}^d denote the pull-back of the Poincaré metric on the unit disc in $\mathbb{C}$ by the map $z \mapsto z^d$.

Definition 3.1.13. A branched hyperbolic structure on Σ is a path metric such that every point x admits a neighbourhood which is locally isometric to a neighbourhood of 0 with the metric $g_{hyp}^{d_x}$ for some $d_x \geq 1$.

The points where $d_x > 1$ are called ramifications and the quantity $d_x - 1$ is the *multiplicity* of the ramification.

There is a well-developed theory of branched structures on surfaces [135]. In particular, branched hyperbolic structures on Σ_g can alternatively be described by the data of:

— A *holonomy representation* $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$,

— A complex structure on Σ_g

— A *developing map* from $\widetilde{\Sigma}_g$ to $\mathbb{H}^2$ which is ρ-equivariant, holomorphic or anti-holomorphic and non-constant (but not necessarily a local diffeomorphism).

The ramification points are then the points where the holomorphic map is non immersive. An important open problem is the following:

Question 3.1.14. *Which representations $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ are the holonomy of a branched hyperbolic structure ?*

We already saw that representations with maximal Euler class are exactly the holonomies of smooth hyperbolic structures. More generally, if h is a branched hyperbolic structure with k branched points (counted with multiplicity) and ρ its holonomy representation, then one has a Gauss–Bonnet formula:

$$\frac{1}{2\pi}\mathbf{Vol}(h) = 2g - 2 - k = |\chi(\rho)| \ ,$$

where $\mathbf{Vol}(h)$ denotes the volume of (Σ_g, h). In particular, the number of branched points cannot exceed $2g - 3$.

There is a lot of flexibility in branched hyperbolic structures: Troyanov [199], generalizing Poincaré–Koebe's uniformization theorem, showed that, given any effective divisor D of degree $k < 2g - 2$ on a Riemann surface of genus g, there exists a unique conformal hyperbolic metric with ramification divisor D. Thus, the space of branched hyperbolic structures has complex dimension $3g - 3 + k$. On the other side, given a hyperbolic surface with k branched points, there is a complex k-dimensional family of deformations of the hyperbolic structure preserving its holonomy. This is consistent with the fact that holonomies of branched hyperbolic structures form an open set in $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$, by a variant of the Ehresmann–Thurston principle.

Not all representations arise this way: Tan [185] noticed that, if f is a continuous map from Σ_g to $\Sigma_{g'}$ that is not homotopic to a branched cover (for instance a map of degree one from Σ_g to Σ_{g-1} that "crushes a handle"), then, for a Fuchsian representation j of $\Gamma_{g'}$, the composition $j \circ f_*$ cannot be the holonomy of a branched hyperbolic structure on Σ_g. However, these examples are not generic (they have non-trivial kernel and discrete image). This lead Goldman to the following conjecture:

Conjecture 3.1.15 (Goldman [78])**.** *Let $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ be an injective representation with non-zero Euler class. Then ρ is the holonomy of a branched hyperbolic metric on Σ_g.*

In an ongoing work with Bertrand Deroin, we obtain partial results towards this conjecture, which will be discussed in Section 3.4.2.

3.1.4 Higher Teichmüller spaces

Higher Teichmüller theory is, in a broad sense, the study of surface group representations into Lie groups of higher dimension, and how they compare to Fuchsian representations. In a more restricted sense, it intends to find and describe analogs of the Teichmüller space inside character varieties of higher rank Lie groups.

Quasi-Fuchsian representations

Historically, the study of differential equations on Riemann surfaces during the second half of the XIX$^{\text{th}}$ century, which culminated with Klein and Poincaré's uniformization theorem, is intimately related to surface group representations into $\mathrm{PSL}(2,\mathbb{C})$.

The character variety of a surface group into $\mathrm{PSL}(2,\mathbb{C})$ has two connected components distinguished by their Stiefel–Whitney class, which vanishes if and only if the representation lifts to $\mathrm{SL}(2,\mathbb{C})$. The component of Stiefel–Whitney class 0 contains an open domain $\mathfrak{X}_{QF}(\Gamma_g, \mathrm{PSL}(2,\mathbb{C}))$ consisting of quasi-isometric embeddings. These quasi-isometric embeddings are called *quasi-Fuchsian representations*.

Recall that, if $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{C})$ is a quasi-Fuchsian representation, then there exists a ρ-equivariant injective continuous map

$$\xi_\rho : \partial_\infty \Gamma_g \simeq \mathbb{S}^1 \to \partial_\infty \mathbb{H}^3 .$$

The complement of the image of ξ_ρ consists of two complex discs $\Omega_\rho^\pm$ on which ρ acts properly discontinuously and cocompactly. This defines a map

$$\begin{array}{rccc} \mathrm{Unif}_{AB} : & \mathfrak{X}_{QF}(\Gamma_g, \mathrm{PSL}(2,\mathbb{C})) & \to & \mathcal{T}_g^+ \times \mathcal{T}_g^- \\ & \rho & \mapsto & \left(\rho(\Gamma_g)\backslash\Omega_\rho^+, \rho(\Gamma_g)\backslash\Omega_\rho^-\right) . \end{array}$$

The Double Uniformization Theorem of Ahlfors–Bers states that this map is a homeomorphism [3].

In contrast with the case of $\mathrm{PSL}(2,\mathbb{R})$, quasi-Fuchsian representations do not form a connected component of $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{C}))$. They are indeed contained in the connected component of the trivial representation. The closure of $\mathfrak{X}_{QF}(\Gamma_g, \mathrm{PSL}(2,\mathbb{C}))$ is the set of discrete and faithful representations, as a consequence of Brock–Canary–Minsky's *ending lamination theorem* [33]. Finally, let us mention that $\mathfrak{X}_{QF}(\Gamma_g, \mathrm{PSL}(2,\mathbb{C}))$ is a maximal domain of proper discontinuity for the action of MCG_g (see [180]).

Hitchin representations

Using the non-Abelian Hodge correspondance, Hitchin proved in [90] that the character variety $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(n,\mathbb{R}))$ ($n \geq 3$) has 3 connected components for odd n and 6 for even n. Interestingly, these components are not distinguished by topological invariants. More precisely, Hitchin proved the existence of one (for odd n) or two (for even n) connected components which do not contain any representation into a compact group, despite having the same topological invariants as some representation into $\mathrm{PSO}(n, \mathbb{R})$.

Definition 3.1.16. An *n-Fuchsian representation* is a representation of Γ_g into $\mathrm{PSL}(n, \mathbb{R})$ of the form $\iota_n \circ j$, where $j : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ is Fuchsian and $\iota_n : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(n, \mathbb{R})$ is the irreducible representation (unique up to conjugation).

The composition with ι_n defines a map from $\mathfrak{X}_{Fuchs}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ to $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(n, \mathbb{R}))$. For odd n this map is 2 to 1 and its image is connected while for n even it is injective and maps the two components of $\mathfrak{X}_{Fuchs}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ to two disjoint connected components of $\mathfrak{X}(\Gamma, \mathrm{PSL}(n, \mathbb{R}))$.

Theorem 3.1.17 (Hitchin [90])**.** *The connected components of $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(n, \mathbb{R}))$ containing $\iota_n \circ \mathfrak{X}_{Fuchs}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ are homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$ and do not contain any representation with relatively compact image.*

These "exotic" components are since named *Hitchin components* and the representations therein are called *Hitchin representations*.

Hitchin suggested that these components were good candidates for a higher rank analog of the Teichmüller space and asked for a geometric interpretation of these representations. For $n = 3$, an answer was already given by Choi–Goldman [43], who showed that Hitchin representations into $\mathrm{PSL}(3, \mathbb{R})$ are exactly the holonomies of convex projective structures on Σ_g (see Section 3.2.3).

In his groundbreaking work [118], Labourie gave a spectacular answer to Hitchin's question in all rank: he introduced the notion of Anosov representation in order to show that Hitchin representations where all Anosov, with additional convexity properties.

Theorem 3.1.18. *Let $\rho : \Gamma_g \to \mathrm{PSL}(n, \mathbb{R})$ be a Hitchin representation. Then ρ is Anosov with respect to a minimal parabolic subgroup. Moreover, the associated boundary map $\xi_\rho : \partial_\infty \Gamma_g \to \mathbb{R}\mathbf{P}^{n-1}$ is Frenet and hyperconvex.*

Without giving a precise definition, let us just say that "Frenet" means that the curve admits a complete osculating flag, and hyperconvex is a very strong transversality property for this family of osculating flags. Labourie's work implies in particular that a Hitchin representation ρ is always discrete and faithful, and maps every $\gamma \in \Gamma_g$ to a diagonalizable matrix with distinct positive eigenvalues.

Hitchin components can be defined more generally for every real *split* semisimple Lie group (such as $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(n, n)$ and $\mathrm{SO}(n+1, n)$), for which both works of Hitchin and Labourie extend.

Maximal representations

Concomitantly to Labourie's work on Hitchin representations, Burger–Iozzi–Wienhard made tremendous progress on the understanding of *maximal representations* into Hermitian Lie groups. In particular, they proved that these representations have some "Fuchsian-like" behaviour, which was interpreted as an Anosov property in [35].

A Lie group G is called *Hermitian* if its symmetric space carries a G-invariant Kähler structure. The associated Kähler form is a "Chern–Weil form" and thus defines a topological invariant of surface group representations into G with values in $\mathbb{Z}$, called the *Toledo invariant.* In the case of $G = \mathrm{PSL}(2,\mathbb{R})$, it coincides with the Euler class. Burger, Iozzi and Wienhard [36] proved a Milnor–Wood inequality for the Toledo invariant and initiated the more systematic study of *maximal representations*, for which this inequality is an equality.

This invariant is named after Domingo Toledo, who introduced it in [197] for $G = \mathrm{SU}(n,1)$. There, he proved some Milnor–Wood inequality and showed that the equality case imposes a remarkable form of rigidity: representations with maximal Toledo invariant take value (up to conjugation) into $\mathrm{U}(1,1) \times \mathrm{U}(n-1)$, and their projection to $\mathrm{PU}(1,1) \simeq \mathrm{PSL}(2,\mathbb{R})$ is Fuchsian. This result was generalized by Burger–Iozzi–Wienhard:

Theorem 3.1.19 (Burger–Iozzi–Wienhard [36])**.** *Let G be a Lie group of Hermitian type and ρ a representation of Γ_g into G. Then*

$$|\mathrm{Tol}(\rho)| \leq (2g-2)\mathbf{rk}(G) \ .$$

Moreover, if $\mathrm{Tol}(\rho) = (2g-2)\mathbf{rk}(G)$, then ρ takes values into a subgroup of G of tube type.

Rather than defining the groups of tube type, let us just say that they are products of simple Hermitian Lie groups of tube type, and that the non-exceptional such groups are isogenous to $\mathrm{SU}(p,p)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(4n)$ or $\mathrm{SO}_0(2,n)$.

Definition 3.1.20. A representation $\rho : \Gamma \to G$ is called *maximal* if

$$\mathrm{Tol}(\rho) = (2g-2)\mathbf{rk}(G) \ .$$

If G is not of tube type, none of the maximal representations are Zariski dense by Theorem 3.1.19. In contrast, when G is of tube type, Burger–Iozzi–Wienhard show the existence of Zariski dense representations with maximal Toledo invariant. Finally, they prove that maximal representations satisfy some Anosov property:

Theorem 3.1.21 (Burger–Iozzi–Labourie–Wienhard [35])**.** *Let $\rho : \Gamma \to G$ be a maximal representation. Then ρ is Anosov with respect to the parabolic subgroup fixing a point in the Shilov boundary.*

In particular, every maximal representation ρ is discrete and faithful, and induces a ρ-equivariant continuous map ξ_ρ from $\partial_\infty \Gamma_g$ to the Shilov boundary of G.

Example 3.1.22.
- — The Shilov boundary of $\mathrm{Sp}(2g, \mathbb{R})$ is the space of Lagrangian subspaces of $\mathbb{R}^{2g}$.
- — The Shilov boundary of $\mathrm{SO}_0(2, d)$ is the space of isotropic lines in $\mathbb{R}^{2,d}$.

The connected components $\mathfrak{X}_{\max}(\Gamma, G)$ of maximal representations can thus be seen as higher rank analogs of the Teichmüller space, in the sense that they consist only of discrete and faithful representations. Interestingly, $\mathfrak{X}_{\max}(\Gamma, G)$ is not always connected, and can have a richer topology than that of the Hitchin components. Even more surprisingly, for $G = \mathrm{PSp}(4, \mathbb{R})$, the set $\mathfrak{X}_{\max}(\Gamma, G)$ has some "exotic" connected components consisting only of Zariski dense representations [30].

The central questions of higher Teichmüller theory

Hitchin an maximal representations in a Lie group G share many properties with Fuchsian representations. First, they are discrete and faithful, and in fact enjoy all the nice propoerties of Anosov representations (limit curves, domains of discontinuity...). Moreover, unlike quasi-Fuchsian representations, they are stable under *any continuous deformation* inside the real group G. This motivates the following definition:

Definition 3.1.23. A *higher Teichmüller space* is a connected component of $\mathfrak{X}(\Gamma, G)$ (for some real semisimple Lie group G) consisting only of discrete and faithful representations.

Higher Teichmüller theory is, in the more restricted sense, the study of higher Teichmüller spaces and the representations therein. It is driven by three fundamental questions that we present below.

Question 3.1.24. *Can we list the higher Teichmüller spaces ?*

This first question is close to being completely answered. Using Higgs bundle techniques, the authors of [8] were able to give a restricted list of candidates: besides Hitchin and maximal components, a third potential family is formed by certain components of the $\mathrm{SO}(p, q)$ character variety. It is extremely likely that these components ar exactly those formed by the Θ-*positive representations* introduced by Guichard–Wienhard (see next question).

Question 3.1.25. *What are the specific properties of the representations in higher Teichmüller spaces ?*

Higher Teichmüller spaces most likely contain only Anosov representations. This, however, does not account for all their specificity, since the Anosov property is not closed in general. In all examples, some additional geometric control on the limit curve of Anosov representations in higher Teichmüller spaces prevent them to "bend towards" a non-discrete representation. (For Hitchin representations, this is the hyper-convexity of the limit curve.) Guichard and Wienhard formilized it into the general notion of Θ-*positivity*, which guaranties for instance a Lipschitz limit set [85]. They are close to proving that Θ-positivity is indeed a closed condition, which would yield a positive answer to the following conjecture:

Conjecture 3.1.26. *Let ρ be a representation of Γ_g into a Lie group G. The following are equivalent:*
— *ρ is Anosov and Θ-positive*
— *every continuous deformation of ρ is discrete and faithful*
— *ρ is either Hitchin or maximal, or $G \simeq \mathrm{SO}(p,q)$ and ρ lies in one of the components of $\mathfrak{X}(\Gamma, G)$ singled out in [8].*

Other geometric and dynamical properties of surface group representations seem related to the Θ-positivity:
— Positivity of associated cross-ratios on $\partial_\infty \Gamma_g$,
— Uniform controls on the length spectrum of the representation,
— Collar lemmas.
These properties will be discussed in the next sections.

Question 3.1.27. *Do higher Teichmüller spaces carry a natural geometry similar to that of $\mathcal{T}_g$?*

The richness of Teichmüller theory stems from the diversity of viewpoints on that space (as a space of complex structures, of hyperbolic metrics or of linear representations). A constant trend in higher Teichmüller theory is to try to diversify the points on view on those higher Teichmüller spaces and enrich their geometry. Let us briefly mention some of them:
— The non-Abelian Hodge correspondance (see Section 3.2.1) answers the above question in a way which is not completely satisfying. Indeed, it endows real character varieties with a structure of complex quasi-projective variety for which the Atiyah–Bott symplectic form is Kähler. however, this additional structure is not quite natural since it depends on the choice of a complex structure on Σ_g. In particular, it is not invariant under the mapping class group. To address this issue, one strategy is to construct a natural MCG_g-invariant projection from a higher Teichmüller space to $\mathcal{T}_g$. This is the motivation behind a conjecture of Labourie presented in Section 3.2.1.

— Labourie, Bridgeman, Canary and Sambarino used tools from hyperbolic dynamics to construct a *pressure metric* on Hitchin components, which extends the Weil–Petersson metric on $\mathcal{T}(\Sigma)$. An interesting question is whether this metric is Kähler for some complex structure. If it is Kähler, though, the associated symplectic form cannot be the Goldman symplectic form, according to the work of Labourie–Wentworth [124].

— Recently, Fock and Thomas introduced the deformation space of *higher complex structures of rank n* on a surface Σ, which has a nice Kähler geometry. They conjecture a natural identification of this space with the $\mathrm{PSL}(n,\mathbb{R})$-Hitchin component [62].

— In Section 3.4.1, we discuss yet another approach initiated in [192], which consists in embedding Hitchin components into the infinite dimensional Teichmüller space of a two dimensional foliation.

3.2 Harmonic maps and applications

Many of my contributions to the study of surface group representations and higher Teichmüller theory have as a common feature the use of harmonic analysis on Riemann surfaces. In this section, we introduce the theory of equivariant harmonic maps and its relation with minimal surfaces and Higgs bundles, before explaining how we applied it to study geometric properties of surface group representations.

3.2.1 Harmonic maps, minimal surfaces, and Higgs bundles

Harmonic maps

Let G be a semisimple Lie group and K a maximal compact subgroup. Denote by $X = G/K$ the symmetric space of G and by g_X its symmetric Riemannian metric.

For a given complex structure J on Σ_g, denote by g_J the unique conformal hyperbolic metric on (Σ_g, J) and by $j : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ the Fuchsian representation uniformizing (Σ_g, J) (defined modulo conjugation). Finally, let ρ be a representation from Γ_g to G, and $f : \widetilde{\Sigma}_g \to X$ a smooth ρ-equivariant map.

The pull-back metric $f^* g_X$ factors to Σ_g and decomposes in the form

$$\varphi_f + e_J(f) g_J + \bar{\varphi}_f ,$$

where $e_J(f)$ is a function called the *energy density* of f (with respect to g_J) and φ_f is a symmetric 2-form of type $(2,0)$ with respect to J, i.e. a smooth section of K_J^2, where K_J denotes the holomorphic cotangent bundle of (Σ_g, J). The form φ_f is called the *Hopf differential* of f.

Definition 3.2.1. The *total energy* of f is the integral of the energy density against the volume form associated to g_J:

$$E_J(f) = \int_\Sigma e_J(f) \mathrm{d}vol_{g_J} \ .$$

The map f is *harmonic* if it is a critical point of E_J.

Remark 3.2.2. For later purposes, we defined the energy density $e_J(f)$ with respect to the uniformizing metric g_J. Changing g_J by a conformal factor e^σ would scale the energy density by $e^{-\sigma}$. However, it would also scale the volume form by e^σ, so that the total energy E_J would remain unchanged. The total energy and the notion of harmonicity are thus inherently conformal notions.

Corlette, building on the work of Eells and Sampson, proved the fundamental existence result for equivariant harmonic maps:

Theorem 3.2.3 (Eells–Sampson [58], Corlette [47]). *If $\rho : \Gamma_g \to G$ is reductive, then there exists a smooth ρ-equivariant harmonic map from $(\widetilde{\Sigma}_g, J)$ to X, which minimizes the energy. Moreover, this harmonic map is unique up to post-composition with an isometry centralizing ρ. In particular, if ρ is irreducible, this harmonic map is unique.*

We will denote by $f_{J,\rho}$ this (essentially unique) ρ-equivariant harmonic map, by $\mathbf{E}(J,\rho)$ its energy and by $\Phi(J,\rho)$ its Hopf differential. One easily verifies that $\mathbf{E}(J,\rho)$ only depends on the isotopy class of J and the conjugacy class of ρ, so that it defines a functional

$$\mathbf{E} : \mathcal{T}_g \times \mathfrak{X}(\Gamma_g, G) \to \mathbb{R}_+ \ .$$

Harmonic maps allow to introduce analytic tools in topology by providing "canonical" maps between Riemannian manifolds in a given homotopy class. They are particularly powerful when the domain is a Riemann surface because they produce holomorphic objects. A first instance of this phenomenon is the following result of Hopf:

Proposition 3.2.4 (Hopf). *The Hopf differential of a harmonic map is holomorphic.*

Higgs bundles

While the notion of energy and harmonicity can be defined more generally for maps between Riemannian manifolds, the particular feature of harmonic maps from Riemann surfaces to symmetric spaces is that the algebraic properties of their differential is encoded in a holomorphic objet called a *Higgs bundle*.

To explain this let us first consider the case where $G = \mathrm{SL}(n,\mathbb{C})$. To every representation $\rho : \Gamma \to G$ is canonically associated a flat complex vector bundle E_ρ of rank n, given by

$$E_\rho = \widetilde{\Sigma}_g \times \mathbb{C}^n/(x,v) \simeq (\gamma\cdot x, \rho(\gamma)v) \ .$$

We denote by ∇_ρ the flat connection on E_ρ. This connection induces a trivialization of the determinent bundle $\det(E_\rho)$ since the representation ρ takes values in $\mathrm{SL}(n,\mathbb{C})$.

Let h be a Hermitian metric on E_ρ whose associated metric on $\det(E_\rho)$ is the trivial metric. Lifting h to $\widetilde{\Sigma}_g$, one can see h as a ρ-equivariant map from $\widetilde{\Sigma}_g$ to the space of Hermitian scalar products on $\mathbb{C}^n$ of fixed determinent, i.e. the symmetric space $X = \mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$. The flat connection cannically decomposes as

$$\nabla_\rho = \nabla_h + \Psi \ ,$$

where ∇_h is a connection preserving h and Ψ a 1-form with values in the h-self-adjoint traceless endomorphisms of E_ρ. The form Ψ can be interpreted as the differential of the equivariant map from $\widetilde{\Sigma}_g$ to X associated to h. In particular, the pull-back of the metric g_X by the equivariant map f_h corresponding to h is given by

$$f_h^* g_X = \mathrm{Tr}(\Psi^2) \ . \tag{3.1}$$

Let us now endow Σ_g with a complex structure J. Then ∇^h and Ψ decompose further into $(1,0)$ and $(0,1)$ types, yielding

$$\nabla_\rho = \bar\partial_h - \bar\partial_h^* + \Theta + \Theta^* \ ,$$

where:

— $\bar\partial_h$ is a *$\bar\partial$-operator*, satisfying the Leibniz rule

$$\bar\partial_h(fs) = (\bar\partial f)s + f(\bar\partial_h s) \ ,$$

— Θ is a $(1,0)$-form with values in $\mathrm{End}(E_\rho)$,

— The $*$ refers to the adjunction with respect to the metric h, so that $\bar\partial_h - \bar\partial_h^* = \nabla_h$ and $\Theta + \Theta^* = \Psi$.

The operator $\bar\partial_h$ defines a holomorphic structure on E_ρ, for which ∇_h is the Chern connection of h. We call the metric h *harmonic* if the corresponding equivariant map from $(\widetilde{\Sigma}_g, J)$ to X is harmonic.

Proposition 3.2.5. *The metric h is harmonic if and only if Θ is holomorphic with respect to $\bar\partial_h$, i.e.*

$$\bar\partial_h \Theta = 0 \ .$$

Remark 3.2.6. This proposition essentially says that h (seen as an equivariant map) is harmonic if and only if the $(1,0)$ part of its differential is holomorphic, which can be seen as a generalization of the well-known fact that real harmonic functions on a Riemann surface are the real part of a holomorphic function.

Remark 3.2.7. By (3.1), the Hopf differential of the harmonic map associated to the harmonic metric h is $\mathrm{Tr}(\Theta^2)$, and we recover the fact that it is holomorphic.

Definition 3.2.8. A $\mathrm{SL}(n,\mathbb{C})$*-Higgs bundle* on (Σ, J) is a pair $(\mathcal{E}, \Theta)$, where $\mathcal{E}$ is a holomorphic vector bundle of rank n such that $\det(\mathcal{E})$ is holomorphically trivial and Θ is a holomorphic 1-form with values in $\mathrm{End}_0(\mathcal{E})$ (where End_0 stands for traceless endomorphisms).

Together with Proposition 3.2.5, Theorem 3.2.3 allows to construct one direction of the *non-Abelian Hodge correspondance*, which associates to a reductive representation ρ the Higgs bundle $((E_\rho, \bar{\partial}_h), \Theta_h)$ where h a harmonic metric on E_ρ. To invert this map, one starts with a $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle $((E, \bar{\partial}_E), \Theta)$ and wants to find a Hermitian metric h such that the connection $\bar{\partial}_E - \bar{\partial}_E^* + \Theta + \Theta^*$ is flat. This boils down to solving the *self-dual equation*

$$F_h + [\Theta, \Theta^*] = 0 \ ,$$

where F_h is the curvature of the Chern connection of h and $*$ is the adjunction with respect to h. A necessary condition (essentially due to S. Kobayashi) is that $((E, \bar{\partial}_E), \Theta)$ should be *polystable* in the following sense:

Definition 3.2.9. A $\mathrm{SL}(n,\mathbb{C})$ Higgs bundle $(\mathcal{E}, \Theta)$ is *stable* if every Θ-invariant holomorphic subbundle (other than $\{0\}$ and $\mathcal{E}$) has negative degree. It is *polystable* if it is a direct sum of stable Higgs bundles of degree 0.

Hitchin (for $n = 2$) and Simpson (in general) proved that the polystability condition is also sufficient:

Theorem 3.2.10 (Hitchin [89], Simpson [178])**.** *A Higgs bundle is polystable if and only if it admits a Hermitian metric h solving the self-duality equations. Moreover, this metric is unique up to an automorphism of the Higgs bundle. In particular, if the Higgs bundle is stable, then it is unique up to scaling.*

Putting these results together, one obtains an equivalence of categories between reductive representations of Γ_g into $\mathrm{SL}(n,\mathbb{C})$ and polystable $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles. This equivalence of category is called the *non-Abelian Hodge correspondence*. It is also a real analytic bijection between the character variety $\mathfrak{X}(\Gamma_g, \mathrm{SL}(n,\mathbb{C}))$ and the moduli space $\mathcal{M}_{Higgs}((\Sigma_g, J), \mathrm{SL}(n,\mathbb{C}))$ of (polystable) $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles over the Riemann surface (Σ_g, J).

Despite its very transcendental nature, some properties transit well through the non-Abelian Hodge correspondence. An important one is that the correspondence "recognizes" representations with values into a real subgroup G.

Let G be a semisimple subgroup of $\mathrm{SL}(n, \mathbb{C})$ and K its maximal compact subgroup. Let $\mathfrak{m}$ denote the orthogonal of $\mathrm{Lie}(K)$ in $\mathfrak{g}$, and let $K_{\mathbb{C}}$ and $\mathfrak{m}_{\mathbb{C}}$ denote the respective complexifications of K and $\mathfrak{m}$.

Theorem 3.2.11 (see [178])**.** *Let ρ be a representation of Γ into $\mathrm{SL}(n, \mathbb{C})$, and $(\mathcal{E}, \Theta)$ the associated Higgs bundle. The following are equivalent:*

— *The representation ρ is conjugate to a representation with values in G,*

— *The structure group of $\mathcal{E}$ admits a holomorphic reduction to $K_{\mathbb{C}}$, for which Θ takes values in the in the associated $\mathfrak{m}_{\mathbb{C}}$-bundle.*

One can derive from this theorem a definition of a *G-Higgs bundle*, and a non-Abelian Hodge correspondence between G-Higgs bundles and representations into G. Let us illustrate this in some examples:

Example 3.2.12. A $\mathrm{SL}(n, \mathbb{R})$-Higgs bundle is a $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle with a holomorphic section Q of $\mathrm{Sym}^2(\mathcal{E}^*)$ which is non-degenerate at every point and such that Θ is self-adjoint with respect to Q.

Example 3.2.13. A $\mathrm{SU}(p, q)$-Higgs bundle is a $\mathrm{SL}(p+q, \mathbb{C})$-Higgs bundle $(\mathcal{E}, \Theta)$ with a holomorphic decomposition $\mathcal{E} = \mathcal{U} \oplus \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ respectively have rank p and q, and such that Θ maps $\mathcal{U}$ into $\mathcal{V}$ and $\mathcal{V}$ into $\mathcal{U}$.

Another property of representations that transit well through the correspondence are the topological invariants associated to a representation. Indeed, they are by definition the topological invariants associated to a K-reduction of the flat principal G-bundle associated to ρ, and thus correspond to topological invariants of the $K_{\mathbb{C}}$-bundle underlying the associated G-Higgs bundle.

Example 3.2.14. Let ρ be a reductive representation into the Hermitian Lie group $\mathrm{SU}(p, q)$ and let $(\mathcal{U} \oplus \mathcal{V}, \Theta)$ be the associated $\mathrm{SU}(p, q)$-Higgs bundle. Then

$$\mathrm{Tol}(\rho) = \deg \mathcal{U} - \deg \mathcal{V} = 2 \deg \mathcal{U} \ .$$

Interestingly, the moduli space $\mathcal{M}_{Higgs}((\Sigma_g, J), G)$ of polystable G-Higgs bundles on (Σ_g, J) is a complex algebraic variety even though the group G is real. When G is complex, the complex structure of the moduli space of Higgs bundles does not agree with the complex structure of $\mathfrak{X}(\Gamma_g, G)$. In fact, these two complex structures fit together in a hyper-Kähler structure. Hitchin [88, 90] and Simpson [178] introduced many tools to study the topology and complex geometry of $\mathcal{M}_{Higgs}((\Sigma_g, J), G)$, from which one derives the remarkable results on the topology of character varieties that have been mentioned in Section 3.1.

The non-Abelian Hodge correspondence has been extended to Riemann surfaces with punctures [177], to higher dimensional compact Kähler manifolds [178], and eventually to complex quasi-projective varieties [26, 148].

Minimal surfaces

We just saw that equivariant harmonic maps associate holomorphic objects to surface group representations, which enhances considerably the structure of their character varieties. A drawback of these constructions is that it depends on the choice of a complex structure on Σ_g, so that this additional structure is in some sense not "natural". In particular, it is not invariant under MCG_g.

In order to remediate this, one might want to find a preferred complex structure associated to a given representation. A general strategy for this consists in minimizing the energy $\mathbf{E}(J, \rho)$ as a function of J. This strategy succeeds in some cases, and is conjectured to work in greater generality.

To be more precise, let us recall first that the cotangent bundle to $\mathcal{T}_g$ at a point J is canonically identified with the space $H^0(K_J^2)$ of holomorphic quadratic differentials on (Σ_g, J).

Let ρ be a reductive representation of Γ_g into a semi-simple Lie group G. The *energy functional* on $\mathcal{T}_g$ associated to ρ is the function

$$\mathbf{E}_\rho : J \mapsto \mathbf{E}(J, \rho) \ .$$

A classical formula (see [203]) states that the differential of the energy functional is given by the Hopf differential of the harmonic map:

Proposition 3.2.15. *At a point $J \in \mathcal{T}_g$ we have*

$$d\mathbf{E}_\rho(J) = -4\Phi(J, \rho) \ .$$

Corollary 3.2.16. *The complex structure J is a critical point of $\mathbf{E}_\rho$ if and only if the harmonic map $f_{J,\rho}$ is conformal, meaning that $f_{J,\rho}^* g_X$ is conformal to g_J or, equivalently, that its Hopf differential vanishes.*

Conformal harmonic maps are *branched minimal immersions* in the following sense: let J be a critical point of the energy functional $\mathbf{E}_\rho$ (so that $f_{J,\rho}$ is conformal). If $\mathbf{E}_\rho(J) = 0$, then $f_{J,\rho}$ is constant (in which case ρ fixes a point in X and $\mathbf{E}_\rho \equiv 0$). Otherwise, away from a finite number of points where the energy density vanishes, $f_{J,\rho}$ is an immersion whose image is a surface in X that locally minimizes the area. Conversely, if $S \subset X$ is a minimal surface, then the inclusion $S \subset X$ is a conformal harmonic map for the conformal structure on S induced by g_X.

These considerations raise the following questions:

Question 3.2.17. *What is the critical locus of $\mathbf{E}_\rho$?*

Question 3.2.18. *Assume ρ is discrete and faithful. Does $\rho(\Gamma_g)\backslash X$ contain a minimal surface diffeomorphic to Σ_g? If so, is this surface unique ?*

Let us now discuss these questions on several examples.

Representations into $\mathrm{PSL}(2,\mathbb{R})$**.**

When ρ takes values into $\mathrm{PSL}(2,\mathbb{R})$, the ρ-equivariant conformal maps from $(\widetilde{\Sigma}_g, J)$ to $X = \mathbb{H}^2$ are either holomorphic or anti-holomorphic. Therefore, finding a critical point of $\mathbf{E}_\rho$ is equivalent to finding a branched hyperbolic structure on Σ_g with holonomy ρ. If J is a critical point, then $\mathbf{E}_\rho(J) = 2\pi|\mathbf{eu}(\rho)|$ is a global minimum of $\mathbf{E}_\rho$, and the locus of such minima, when non-empty, is a holomorphic submanifold of $\mathcal{T}_g$ of dimension $2g - 2 - |\mathbf{eu}(\rho)|$.

Quasi-Fuchsian representations.

Let now ρ be a quasi-Fuchsian representation of Γ_g into $\mathrm{PSL}(2,\mathbb{C})$. One can think of ρ-equivariant minimal embeddings as solutions of an asymptotic Plateau problem with boundary given by the limit set of ρ in $\partial_\infty \mathbb{H}^3$.

Schoen–Yau [173] and Sacks–Uhlenbeck [168] proved the properness of the energy functional $\mathbf{E}_\rho$ and deduced the existence of ρ-equivariant conformal harmonic maps, and Freedman–Hass–Scott [63] proved that the conformal harmonic map corresponding to the global minimum of $\mathbf{E}_\rho$ is an embedding.

It seems that the finiteness of the critical locus of $\mathbf{E}_\rho$ is still unproven. One can show nevertheless that this set is compact, so that proving its finiteness boils down to proving that every minimal surface in a quasi-Fuchsian manifold is locally rigid. Huang and Wang proved that some quasi-Fuchsian representations can have arbitrarily many equivariant minimal embeddings [91]. However, when the principal curvatures of a minimal surface are bounded by 1, Uhlenbeck proved that this minimal surface is unique [200]. The corresponding representations are called *almost Fuchsian*.

A heuristic picture behind these (non-)uniqueness results is that the minimal surface should be unique when the limit curve of ρ does not "wind" too much. Seppi proved for instance that a control on the regularity of the limit set of ρ implies a bound on principal curvatures of a minimal surface [175], while the counter-examples to uniqueness of Huang and Wang come from a limit set that approximates a highly meandering curve.

Higher Teichmüller spaces

The fact that the limit set of a quasi-Fuchsian representation can be highly irregular is related to the fact that quasi-Fuchsian representations can "degenerate" to representations which are not quasi-isometric embeddings. This typically does not happen for Hitchin representations and more generally for Guichard–Wienhard's Θ-positive representations, whose limit curves are Lipschitz. It is thus tempting to formulate the following conjecture:

Conjecture 3.2.19. *Let $\rho : \Gamma \to G$ be a Θ-positive representation. Then the energy functional $\mathbf{E}_\rho$ has a unique critical point, whose corresponding conformal harmonic map is an embedding.*

A particular case of the conjecture is a theorem of Schoen stating that, given g_1 and g_2 two hyperbolic metrics on Σ, there is a unique diffeomorphism from (Σ, g_1) to (Σ, g_2) homotopic to the identity whose graph is a minimal surface [172].

The more general conjecture was formulated by Labourie in [120] for Hitchin representations, where he also proved the properness of $\mathbf{E}_\rho$ for any Anosov representation. It was proven independently by Labourie [119] and Loftin [133] for Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$ where minimal surfaces are related to *affine spheres* (see Section 3.2.3). Labourie later extended his result to Hitchin representations into all split real Lie groups of rank 2 (namely, $\mathrm{PSL}(3,\mathbb{R})$, $\mathrm{Sp}(4,\mathbb{R})$ and G_2) [121]. Arguments in favor of the general conjecture came with our work with Brian Collier and Jérémy Toulisse [45], which proved that the conjecture holds for all maximal representations into Hermitian Lie groups of rank 2 (see Section 3.2.4).

The main motivation for Conjecture 3.2.19 is to provide higher Teichmüller spaces with a natural complex structure. To explain this, consider, inside $\mathcal{T}_g \times \mathfrak{X}(\Gamma_g, G)$, the subset $\mathcal{M}_{\min}(\Sigma_g, G)$ consisting of pairs (J, ρ) such that the ρ-equivariant harmonic map $f_{J,\rho}$ is conformal with respect to J. The space $\mathcal{M}_{\min}(\Sigma_g, G)$ is thus the moduli space of branched equivariant mnimal immersions. The non-Abelian Hodge correspondence identifies $\mathcal{T}_g \times \mathfrak{X}(\Gamma_g, G)$ with the space of (equivalence classes of) triples $(J, \mathcal{E}, \Theta)$, where J is a complex structure on Σ_g and $(\mathcal{E}, \Theta)$ is a G-Higgs bundle on (Σ_g, J). This space has a MCG_g-invariant structure of complex analytic variety (see [179, 6]), and the MCG_g-invariant subspace $\mathcal{M}_{\min}(\Sigma_g, G)$ is holomorphic since it is defined by the equation $\mathrm{Tr}(\Theta^2) = 0$ (see Remark 3.2.7).

If we now consider a component of representations for which we know the existence and uniqueness of a minimal surface, then the projection from $\mathcal{M}_{\min}(\Sigma_g, G)$ to the character variety is bijective in restriction to this component, and one can push forward the complex structure.

3.2.2 Application 1: representations into Lie groups of rank 1

I now present various applications of the theory of equivariant harmonic maps to the study of refined properties of surface group representations.

My first application is a "domination theorem" for representations into Lie groups of rank 1. More generally, let X be a complete simply connected Riemannian manifold of sectional curvature bounded above by -1 and let ρ be a representation of Γ into $\mathrm{Isom}(X)$. [1] We call ρ *Fuchsian* if it preserves a totally geodesic plane of curvature -1 in X and acts properly discontinuously on it.

Definition 3.2.20. A Fuchsian representation $j : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ *dominates* ρ if there exists a (j, ρ)-equivariant map from $\mathbb{H}^2$ to X which is λ-Lipschitz for some $\lambda < 1$.

Equivalently, by a theorem of Guéritaud–Kassel [81], j dominates ρ if there exists $\lambda < 1$ such that

$$L_\rho < \lambda L_j$$

where L_ρ and L_j denote respectively the length spectrum of ρ and j acting on X and $\mathbb{H}^2$. We denote by $\mathrm{Dom}(\rho)$ the domain of $\mathcal{T}_g$ formed by conjugacy classes of Fuchsian representations that dominate ρ.

Theorem 3.2.21. *Let ρ be a representation of Γ into $\mathrm{Isom}(X)$. Then, either ρ is Fuchsian or $\mathrm{Dom}(\rho)$ is non-empty and homeomorphic to $\mathcal{T}_g$.*

The non-emptiness of $\mathrm{Dom}(\rho)$ was obtain with Bertrand Deroin [55], and independently by Guéritaud–Kassel–Wolff [82] for representations ρ into $\mathrm{PSL}(2, \mathbb{R})$). The topology of $\mathrm{Dom}(\rho)$ was described in my subsequent work [189].

The primary motivation of these results was their application to the description of the moduli space of anti-de Sitter 3-manifolds, which is discussed in Chapter 2, Section 2.3.5. However, it also has other consequences which are interesting in their own way. The non-emptiness of $\mathrm{Dom}(\rho)$, in particular, allows to extend certain uniform controls on Fuchsian representations to all representations in rank 1.

Corollary 3.2.22. *Let ρ be a representation of Γ_g into $\mathrm{Isom}(X)$. Then:*

- *There exists a simple closed curve $\gamma \in [\Gamma_g]$ such that*

$$L_\rho(\gamma) \leq \sigma_g \ ,$$

  where σ_g is the largest systole in genus g.
- *There exists a pair of pants decomposition given by curves $(\gamma_i) \in [\Gamma]^{3g-3}$ such that*

$$L_\rho(\gamma_i) \leq B_g \ ,$$

  where B_g is the Bers constant in genus g. for all i.

1. Importantly, in this section, we normalize the metric of a symmetric space of rank 1 so that its sectional curvature is everywhere ≤ -1.

The Bers constant in genus 2 was computed in [66] and its extension to all representations in $\mathrm{PSL}(2,\mathbb{R})$ was used as a starting point in [136] to prove Bowditch's conjecture in genus 2.

As another application, one recovers Bowen's entropy rigidity result: If $\rho : \Gamma_g \to \mathrm{Isom}(X)$ is discrete and faithful, then

$$\mathscr{H}(\rho) \geq 1 ,$$

with equality if and only if ρ is Fuchsian.

Finally Theorem 3.2.21 can be applied to a Fuchsian representation ρ acting on $\mathbb{H}^2$ endowed with the metric $e^{-R}h_{\mathbb{H}^2}$, $R > 0$. As a corollary, one obtains a description of the left balls for Thurston's asymmetric distance:

Corollary 3.2.23. *Let j_2 be a point in $\mathcal{T}_g$. For every $R > 0$, the set*

$$B_l(j_2, R) = \{j_1 \in \mathcal{T}_g \mid d_{Th}(j_1, j_2) < R\}$$

is homeomorphic to an open ball in $\mathcal{T}_g$.

Note that the *right balls*

$$B_r(j_2, R) = \{j_2 \in \mathcal{T}(\Sigma) \mid d_{Th}(j_2, j_1) < R\}$$

are convex for the Weil–Petersson distance as a consequence of [207]. This does not readily imply Corollary 3.2.23, though, since d_{Th} is not symmetric.

Let us now explain how the proof of Theorem 3.2.21 uses harmonic maps. Fix a complex structure J on Σ_g. recall that $\Phi(J,\rho)$ denotes the Hopf differential of the ρ-equivariant harmonic map $f_{J,\rho}$ from $(\widetilde{\Sigma}_g, J)$ to X. By a theorem of Wolf [205] – which also follows from Hitchin's Higgs bundle description of $\mathfrak{X}_{Fuchs}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ – there exists a unique Fuchsian representation $j : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ such that $\Phi(J,j) = \Phi(J,\rho)$. Using a maximum principle, we prove in [55] that the (j,ρ)-equivariant map $f_{J,\rho} \circ f_{J,j}^{-1}$ is either isometric and totally geodesic, or contracting. The key to this maximum principle is that harmonic maps are in some sense "saddle shaped" so that the curvature of the pull-back metric $f_{J,\rho}^* g_X$ is bounded everywhere by -1.

Since the whole construction depends on the complex structure J, we obtain a map Ψ_ρ from $\mathcal{T}_g$ to $\mathrm{Dom}(\rho)$. To invert this map, one wants, given a representation j, to find a complex structure J such that

$$\Phi(J,j) = \Phi(J,\rho) .$$

By Proposition 3.2.15, this is equivalent to finding a critical point of $\mathbf{E}_j - \mathbf{E}_\rho$. In [189], I prove that, when j dominates ρ, the functional $\mathbf{E}_j - \mathbf{E}_\rho$ admits a unique critical point. Thus, the map Ψ_ρ gives the required homeomorphism.

Note that the proof associates to a dominating pair (j, ρ) a canonical choice of complex structure J (the critical point of $\mathbf{E}_j - \mathbf{E}_\rho$) and a canonical (j, ρ)-equivariant contracting map from $\mathbb{H}^2$ to X (the map $f_{J,\rho} \circ f_{J,j}^{-1}$). This can be interpreted as a uniqueness result for maximal spacelike surfaces in some pseudo-Riemannian setting: Endow the space $\mathbb{H}^2 \times X$ with the pseudo-Riemannian metric $g_{\mathbb{H}^2} \oplus -g_X$, acted on by Γ_g via the representation $j \times \rho$. The graph of a (j, ρ)-equivariant contracting map is a Γ_g-invariant spacelike disc in $\mathbb{H}^2 \times X$. The canonical contracting map is the unique one whose graph is *maximal* (i.e. has maximal area modulo Γ_g), and the complex structure J is given by the conformal class of the pseudo-Riemannian metric restricted to this maximal disc.

3.2.3 Application 2: Convex $\mathbb{R}\mathbf{P}^2$-structures

Recall that a *proper convex domain* in $\mathbb{R}\mathbf{P}^2$ is an open domain which is convex and bounded in an affine chart. Such domains can be endowed their *Hilbert distance*, which proves to be a useful tool to study discrete group actions on projecive spaces (see Section 1.3.3).

Let now ρ be a Hitchin representation of Γ_g into $\mathrm{PSL}(3, \mathbb{R})$. By a theorem of Choi and Goldman [43], ρ acts properly discontinuously and cocompactly on a proper convex domain $\Omega_\rho \subset \mathbb{R}\mathbf{P}^2$. The *Hilbert length spectrum* of ρ then coincides with its *highest weight length spectrum* (see Section 1.3.3), which we simply denote here by L_ρ.

By the work of Labourie and Loftin [133, 119], the energy functional $\mathbf{E}_\rho$ on $\mathcal{T}_g$ has a unique critical point. In [190], I prove the following comparison result:

Theorem 3.2.24. *Let ρ be a Hitchin representation of Γ into* $\mathrm{PSL}(2, \mathbb{R})$*, and let j be the Fuchsian representation corresponding to the unique critical point of $\mathbf{E}_\rho$. Then either $\rho = \iota_3 \circ j$ or there exists $\lambda > 1$ such that*

$$L_\rho \geq \lambda L_j \ .$$

As a corollary, one obtains a sharp version of the collar lemma of Lee–Zhang [129] in rank 3:

Corollary 3.2.25. *Let $\gamma, \eta \in [\Gamma]$ be two essentially intersecting closed curves on Σ_g. Then*

$$\sinh\left(\frac{L_\rho(\gamma)}{2}\right) \cdot \sinh\left(\frac{L_\rho(\eta)}{2}\right) > 1 \ .$$

The proof of this theorem uses an auxiliary Riemannian metric h_B called the *Blashke metric*. This metric comes from the theory of affine spheres. Very briefly, for every proper convex Ω domain in $\mathbb{R}\mathbf{P}^n$, one can find a canonical smooth convex hypersurface in the cone of Ω in $\mathbb{R}^{n+1}$ asymptotic to its boundary, called the *affine sphere* (which solves a certain Monge-Ampère equation). The affine second fundamental form of this affine sphere is the Blaschke metric. Calabi [38] proved that the Ricci curvature of the Blaschke metric is non-positive and at least $-(n-1)h_B$.

When $\Omega_\rho \subset \mathbb{R}\mathbf{P}^2$ is invariant under a Hitchin representation ρ, the affine sphere and the Blaschke metric are invariant under $\rho(\Gamma)$ and are related to minimal surfaces in the following way: the conformal class of the Blaschke metric is the unique critical point of the energy functional $\mathbf{E}_\rho$ on $\mathcal{T}_g$, and the associated conformal harmonic map to the symmetric space of $\mathrm{PSL}(3, \mathbb{R})$ is some sort of "Gauss map" of the affine sphere [133, 119].

The proof of the inequality $L_j \leq L_\rho$ can now be broken into two pieces. On one side, using a classical maximum principle together with the curvature bound of Calabi, one shows that the Blaschke metric is everywhere greater than the conformal hyperbolic metric. One the other side, we show that the length spectrum of ρ with respect to the Blaschke metric is less than the length spectrum with respect to the Hilbert metric. This last point comes from the following lemma:

Lemma 3.2.26. *Let Ω be a proper convex domain of $\mathbb{R}\mathbf{P}^n$ and let d_H and d_B denote respectively the Hilbert and Blaschke distances on Ω. Then, for all $x, y \in \Omega$, we have*

$$d_B(x,y) < d_H(x,y) + 1 \ .$$

Remark 3.2.27. Lemma 3.2.26 is also at the core of my theorem on volume growth of Hilbert geometries (Theorem 1.3.17).

3.2.4 Application 3: maximal representations in rank 2

Our investigation of higher Teichmüller spaces in relation to minimal surfaces was pursued in a joint work with Brian Collier and Jérémy Toulisse, where we generalize many of the results of the previous section to maximal representations in rank 2.

Recall first that, by work of Burger–Iozzi–Wienhard, maximal representations take value into Hermitian Lie groups *of tube type*. In rank 2, this essentially reduces our study to the familly $G = \mathrm{SO}_0(2, d)$, $d \geq 2$. Note in particular the exceptional isomorphisms

$$\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{PSO}_0(2, 2) \ ,$$

$$\mathrm{PSp}(4, \mathbb{R}) \simeq \mathrm{SO}_0(2, 3) \ ,$$

$$\mathrm{PU}(2, 2) \simeq \mathrm{PSO}_0(2, 4) \ .$$

We prove the following results:

Theorem 3.2.28 (Collier–Tholozan–Toulisse [45])**.** *Let ρ be a maximal representation of Γ_g into $\mathrm{SO}_0(2, d)$. Then the energy functional $\mathbf{E}_\rho$ has a unique critical point. Moreover the associated equivariant conformal harmonic map into the symmetric space of $\mathrm{SO}_0(2, d)$ is an embedding.*

Theorem 3.2.29 (Collier–Tholozan–Toulisse [45]). *Let ρ be a maximal representation of Γ_g into* $\mathrm{SO}_0(2,d)$*. Let j be the Fuchsian representation corresponding to the unique critical point of $\mathbf{E}_\rho$. Then*

$$L_j \leq L_\rho \ .$$

Here, L_ρ denotes again the highest weight length spectrum of a representation $\rho : \Gamma_g \to \mathrm{SO}_0(2,d) \subset \mathrm{SL}(d+2,\mathbb{R})$.

As a corollary, we obtain a sharp collar lemma identical to Corollary 3.2.25 for maximal representations into $\mathrm{SO}_0(2,d)$, as well as an upper bound on their highest weight entropy:

Corollary 3.2.30. *Let ρ be a maximal representation of Γ_g into* $\mathrm{SO}_0(2,d)$*. Then its highest weight entropy satisfies*

$$\mathscr{H}(\rho) \leq 1 \ ,$$

with equality if and only if ρ is Fuchsian.

The central idea of our work is to consider maximal representations into $\mathrm{SO}_0(2,d)$ acting on the pseudo-hyperbolic space $\mathbb{H}^{2,d-1}$. With this point of view, ρ-equivariant *maximal spacelike embeddings* into $\mathbb{H}^{2,d-1}$ play the role of affine spheres in the previous section. We prove the following:

(1) Every critical point of $\mathbf{E}_\rho$ corresponds to a ρ-equivariant maximal spacelike embedding of $\widetilde{\Sigma}_g$ into $\mathbb{H}^{2,d-1}$, the Gauss map of which gives a minimal embedding into the symmetric space of $\mathrm{SO}_0(2,d)$.
(2) There is a unique such maximal spacelike embedding.
(3) The metric induced on Σ_g by this maximal spacelike embedding has curvature greater or equal to -1, and its length spectrum is less or equal to the length spectrum of ρ.

The point (1) follows from interpreting the cyclic structure of the Higgs bundle associated to ρ on (Σ_g, J) when J is a critical point of $\mathbf{E}_\rho$. Point (2) is a variation on an argument given by Bonsante and Schlenker for maximal spacelike surfaces in $\mathbb{H}^{2,1}$. Roughly speaking, one applies a maximum principle to the "height" of a maximal surface with repsect to another one. This makes sense because the pseudo-Riemannian nature of the metric imposes a strong geometric control on spacelike manifolds of maximal dimension. Finally, Point (3) follows from classical inequalities for minimal surfaces in hyperbolic spaces which are all reversed here because of the mixed signature.

Let me finally mention a further application of our work: Recall that maximal representations $\rho : \Gamma_g \to \mathrm{SO}_0(2,d)$ are Anosov with respect to the stabilizer of an isotropic line. The associated limit curve is the boundary at infinity of our ρ-invariant spacelike disc $\mathcal{D}$ in $\mathbb{H}^{2,d-1}$. Now the work of Guichard–Wienhard [84] associates to ρ a cocompact domain of discontinuity Ω_ρ in the space of totally isotropic planes in $\mathbb{R}^{2,d}$, which consists of all isotropic 2-planes that do not intersect the limit curve. In [45], we show that every totally isotropic plane in Ω_ρ is orthogonal to a unique point in $\mathcal{D}$. This defines a ρ-equivariant geometric fibration from Ω_ρ to $\mathcal{D}$ and allows us to describe the topology of $\rho(\Gamma)\backslash\Omega_\rho$, which is not accessible with Guichard–Wienhard's construction.

3.2.5 Conjectural properties of higher Teichmüller spaces

The uniform controls on the length spectrum of Hitchin representations into $\mathrm{PSL}(3,\mathbb{R})$ and maximal representations in rank 2 that were presented in the previous sections motivate the following general conjecture:

Conjecture 3.2.31. *Let $\widehat{\mathcal{T}} \subset \mathfrak{X}(\Gamma_g, G)$ be a Higher Teichmüller space. Then there exists a constant $C > 0$ such that, for any $\rho \in \widehat{\mathcal{T}}$, there exists a Fuchsian representation j such that*

$$L_\rho \geq C L_j \ .$$

Remark 3.2.32. Here L_ρ denotes the length spectrum for some metric on the symmetric space of G and the constant C depends on the choice of this metric. In practice, for each higher Teichmüller space, one could hope for a sharp result with a suitable choice of length spectrum.

As in the previous sections, the conjecture would imply that the entropy of ρ (for the corresponding choice of length spectrum) is bounded above by $\frac{1}{C}$. In fact, Theorem 8 provides a crude upper bound for the entropy of Anosov representations, since the Hausdorff dimension of their limit curve is at most the dimension of the flag variety they live in. In [158], Potrie–Sambarino have given a sharp bound on the highest weight entropy of Hitchin representations in $\mathrm{SL}(n,\mathbb{R})$, and the recent work of Pozzetti–Sambarino–Wienhard [159] implies that the simple weight entropy of any Θ-positive representation is at most 1.

The conjecture would also imply a collar lemma for all representations in $\widehat{\mathcal{T}}$. Such a collar lemma was proven by Lee–Zhang for Hitchin representations [129], by Burger–Pozzetti for maximal representations [37], and recently by Pozzetti–Beyrer for Θ-positive representations [25]. In a forthcoming work, we will prove a collar lemma for all length spectra associated to *positive cross-ratios.*

3.3 Bounded relative character varieties

In this short section, I present my recent works with Bertrand Deroin [56] and Jérémy Toulisse [193], which construct compact relative components in relative character varieties of punctured spheres into Lie groups of Hermitian type. These results will be put in persective in Section 3.4.3.

3.3.1 Relative character varieties into $\mathrm{PSL}(2,\mathbb{R})$

Bertrand and I discovered the existence of these bounded relative components while trying to design a notion of Euler class for representations of surface groups with boundaries with nice additive properties.

The *Toledo invariant*, which can be defined for surfaces with boundary using relative cohomology (see [36]), already provides a good generalization of the Euler class to surfaces with boundary, but a drawback is that it does not take integral values anymore. Adding a correction term given by (a determination in the interval $[0, 2\pi]$ of) the rotation numbers of the images of boundary curves, one obtains an integral valued invariant which we call the *relative Euler class* for representations $\rho : \Gamma_{g,n} \to \mathrm{PSL}(2,\mathbb{R})$.

By a topological recurrence, we proved that the relative Euler class satisfies the Milnor–Wood inequality for surfaces of positive genus. In contrast, for an n-punctured sphere, the Euler class can take the value $n-1$. This singles out relative components which we called *supra-maximal*. We proved in [56] that these components have remarkable properties.

Theorem 3.3.1 (Deroin–Tholozan [56])**.** *Let $a_i, 1 \leq i \leq n$ be rotations of angle $2\pi - \alpha_i$ in $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Isom}_+(\mathbb{H}^2)$. If $0 < \alpha_i < 2\pi$ and $\sum_{i=1}^n \alpha_i < 2\pi$, then the subset of $\mathfrak{X}_{\mathbf{a}}(\Gamma_{0,n}, \mathrm{PSL}(2,\mathbb{R})$ of relative Euler class $n-1$ is a compact connected component symplectomorphic to $\mathbb{C}\mathbf{P}^{n-3}$ with a multiple of the Fubini–Study symplectic form. Moreover the representations ρ in these components satisfy the following properties:*

(1) The $\mathrm{MCG}_{0,n}$-orbit of $[\rho] \in \Omega$ is bounded,

(2) The image by ρ of any simple closed curve has all its eigenvalues of module 1,

(3) For every complex structure J on $\Sigma_{0,n}$, there exists a ρ-equivariant holomorphic map from $(\widetilde{\Sigma}_{0,n}, J)$ to $\mathbb{H}^2$.

Remark 3.3.2. The second property contrasts with a lemma of Gallo–Kapovich–Marden for closed surface group representations into $\mathrm{PSL}(2,\mathbb{C})$ [64].

3.3.2 Higher rank Hermitian Lie groups

Concomitantly to my work with Bertrand, Gabriele Mondello essentially classified the topology of relative character varieties of punctured surfaces into $\mathrm{PSL}(2,\mathbb{R})$ using the parabolic non-Abelian Hodge correspondence [150]. He could in particular recover our results using parabolic Higgs bundles. This approach motivated Jérémy Toulisse and I to search for compact relative components into the moduli spaces of parabolic Higgs bundles for higher rank

Lie groups. We managed to prove their existence for most of the classical Lie groups of Hermitian type:

Theorem 3.3.3 (Tholozan–Toulisse [193])**.** *Let G be one of the Lie groups $\mathrm{PU}(p,q)$, $\mathrm{Sp}(2k,\mathbb{R})$ or $\mathrm{SO}^*(2k)$. For all $n \geq 4$, there exists an open set $\Omega \subset \mathfrak{X}(\Gamma_{0,n}, G)$ which is a union of compact connected components of relative character varieties. Moreover, the representations ρ in Ω have the following properties:*

(1) The $\mathrm{MCG}_{0,n}$-orbit of $[\rho] \in \Omega$ is bounded,
(2) The image by ρ of any simple closed curve has all its eigenvalues of module 1,
(3) For every complex structure J on $\Sigma_{0,n}$, there exists a ρ-equivariant holomorphic map from $(\widetilde{\Sigma}_{0,n}, J)$ to the symmetric space of G.

Though the proof of this Theorem is less constructive than that of Theorem 3.3.1, for some specific choices of parameters, we gave a nice description of these compact components as certain quiver varieties.

These results give an idea of what could be the general properties of bounded mapping class group orbits in (relative) character varieties. This opens a research project which is discussed in Section 3.4.3.

3.4 Research perspectives

To conclude this chapter, I would like to mention three of my ongoing research projects which pursue the study of surface group representations in several different directions. Section 3.4.1 presents a project which aims at reproducing the framework of Teichmüller theory in an infinite dimensional space which "contains" all higher Teichmüller spaces. In Section 3.4.2, I mention partial results toward Goldman's conjecture on branched hyperbolic metrics (Conjecture 3.1.15). Finally, in Section 3.4.3, I describe a conjectural picture of bounded mapping class group orbits which puts in perspective the results presented in the previous section.

3.4.1 Highest Teichmüller theory

As mentioned previously, a constant preoccupation of higher Teichmüller theorists is to extend the beautiful geometry of the Teichmüller space to higher Teichmüller spaces. Though the approach via Higgs bundles via Labourie's conjecture is promissing, there has been very little progress in this direction beyond the rank 2 case discussed above.

In my lecture notes [192], I propose a new approach which consists in embedding higher Teichmüller spaces into and infinite dimensional deformation space of dynamical systems. This bears similarities with Labourie's work

[119] and is strongly related to Bridgeman–Canary–Labourie–Sambarino's construction of the *pressure metric* on Hitchin components [31, 32]. The primary interest of my approach is to propose three different constructions of a *highest Teichmüller space*, one of which does carry a "Teichmüller geometry". My hope is that the geometric features of this highest Teichmüller space restrict nicely to higher Teichmüller components. Though this last step seems the hardest, I believe that the study of the geometry of this highest Teichmüller space is interesting in its own way.

The results of this section are partly prospective. The framework, presented in [192], is not entirely new and consists in gathering works of Bowen [29], Margulis [140], Sullivan [183] or Ledrappier [127]. I will also mention some future work which investigates further the geometry of these higher Teichmüller spaces.

In all this section, we fix Σ a closed surface of genus $g \geq 2$, which we equip with an arbitrary hyperbolic metric. We denote by $T_1\Sigma$ its unit tangent bundle, and by φ the geodesic flow on $T_1\Sigma$. Recall that the set of closed orbits of φ (with multiplicity), is the set of closed geodesics and identifies with the set $[\Gamma]$ of non-trivial conjugacy classes in Γ.

Reparametrizations of the geodesic flow

The first and least original of our highest Teichmüller spaces is the space of continuous reparametrizations of the geodesic flow φ of entropy 1. The description of it we give here seems to date back to the work of Bowen.

A *continuous reparametrization* of φ is a flow ψ on $T_1\Sigma$ with the same orbits and same orientation as φ. One can associate to ψ its *period map*:

$$L_\psi : [\Gamma] \to \mathbb{R}_{>0}$$

which associates to every closed geodesic the time taken by ψ to run through it. The period map almost characterizes the conjugacy class of ψ:

Theorem 3.4.1 (Bowen, Livsic [132]). *If two reparametrizations ψ_1 and ψ_2 have the same period map, then ψ_2 is a uniform limit of conjugates of ψ_1. If moreover ψ_1 and ψ_2 are Hölder continuous, then they are conjugate.*

By work of Bowen, the topological entropy $\mathscr{H}(\psi)$ of a reparametrization ψ equals the exponential growth rate of its period map:

$$\mathscr{H}(\psi) = \lim_{R\to+\infty} \frac{1}{R} \log \sharp\{[\gamma] \in [\Gamma] \mid L_\psi(\gamma) \leq R\} .$$

The topological entropy gives a way to "normalize" a reparametrization: if ψ has entropy λ, then the *scaled flow*

$$\psi^\lambda(x,t) = \psi(x, \lambda t)$$

has entropy 1.

Let us introduce the following spaces:

— The space $\mathrm{Par}(\Sigma)$ of reparametrizations of the geodesic flow modulo the equivalence relation

$$\psi_1 \sim \psi_2 \Longleftrightarrow L_{\psi_1} = L_{\psi_2} \ ,$$

— The subspace $\mathrm{Par}_1(\Sigma)$ of reparametrizations of entropy 1,
— The projectivized space $\mathbf{P}\mathrm{Par}(\Sigma)$ of reparametrizations up to scaling,
— The subset $\mathrm{Par}_1^h(\Sigma)$ of Hölder reparametrizations.

Definition 3.4.2. Our first *highest Teichmüller space* is the space $\mathrm{Par}_1(\Sigma)$, which is canonically homeomorphic to $\mathbf{P}\mathrm{Par}(\Sigma)$.

The space $\mathbf{P}\mathrm{Par}(\Sigma)$ has the structure of an infinite dimensional Banach manifold and carries a natural complete Finsler metric. More precisely, $\mathrm{Par}(\Sigma)$ identifies with the positive cone in the Banach space of cohomology classes of continuous cocycles along φ, and its projectivisation is thus an infinite dimensional convex projective domain which can be endowed with its Hilbert distance d_{Hilb}. In [192], I remark that this Hilbert distance is the analog of the symmetrized Thurston distance on $\mathcal{T}_g$:

Proposition 3.4.3. *Given* $[\psi_1]$ *and* $[\psi_2] \in \mathbf{P}\mathrm{Par}(\Sigma)$, *we have*

$$d_{Hilb}(\psi_1, \psi_2) = \frac{1}{2}\left(\sup_{\gamma\in[\Gamma]} \log\left(\frac{L_{\psi_2}(\gamma)}{L_{\psi_1}(\gamma)}\right) + \sup_{\gamma\in[\Gamma]} \log\left(\frac{L_{\psi_1}(\gamma)}{L_{\psi_2}(\gamma)}\right)\right) \ .$$

The set $\mathrm{Par}_1(\Sigma)$ is a convex hypersurface in $\mathrm{Par}(\Sigma)$ which intersects every ray at a single point. The thermodynamical formalism, developed by Ruelle [167] and many others, gives the main properties of the entropy functional in restriction to Hölder reparametrizations:

Theorem 3.4.4. *The set* $\mathrm{Par}_1^h(\Sigma) = \mathrm{Par}_1(\Sigma) \cap \mathrm{Par}^h(\Sigma)$ *is smooth in* $\mathrm{Par}^h(\Sigma)$ *with positive definite radial second fundamental form.*

Definition 3.4.5. The radial second fundamental form of $\mathrm{Par}_1(\Sigma)$ (when defined) is called the *pressure metric*.

Here, the *radial second fundamental form* refers to the second fundamental form of $\mathrm{Par}_1(\Sigma)$ when taking the radial vector field as a basis for the normal bundle to $\mathrm{Par}_1(\Sigma)$. [2]

It is a well-known fact that the geodesic flow of any negatively curved metric on Σ is conjugate to a reparametrization of the geodesic flow on our fixed hyperbolic structure. This has two consequences:

2. Though the hypersurface of entropy 1 does not solve (to my knowledge) any sort of Monge–Ampère equation, there is some analogy here with the theory of affine spheres. In fact, Lemma 3.2.26, which does not use any analytic property of the affine sphere, applies here and gives a comparison between the pressure and symmetrized Thurston distances.

— The construction and geometry of $\mathrm{Par}(\Sigma)$ are in fact independent of the choice of a hyperbolic structure on Σ, and are thus preserved by an action of the mapping class group of Σ,
— The Teichmüller space $\mathcal{T}(\Sigma)$ (and more generally the space of negatively curved metrics, see [154]) embeds in $\mathrm{Par}_1^h(\Sigma)$.

Moreover, McMullen proved in [144] that the restriction of the pressure metric to the image of $\mathcal{T}(\Sigma)$ coincides with (a multiple of) the Weil–Petersson metric.

More generally, one can associate to an Anosov representation $\rho : \Gamma_g \to G$ certain Hölder reparametrizations of the geodesic flow ψ_ρ whose period map satisfies

$$L_{\psi_\rho}(\gamma) = \alpha(\lambda(\rho(\gamma)))$$

For certain linear forms α on the Weyl chamber of G. (Here, λ denotes the Jordan projection.) Precisely which such forms α give rise to reparametrizations of the geodesic flow an when the corresponding map $\rho \mapsto [\psi_\rho]$ is an immersion depends greatly on the Anosov property and the structure of the group G. Two particularly interesting cases have been studied by Bridgeman–Canary–Labourie–Sambarino:

Theorem 3.4.6 (Bridgeman–Canary–Labourie–Sambarino [31, 32])**.** *There exists an embedding*

$$I_n : \mathrm{Hit}_n \to \mathbf{P}\mathrm{Par}^h(\Sigma)$$

such that for all $\rho \in Hit_n$ and all $\gamma \in [\Gamma]$,

$$L_{I_n(\rho)}(\gamma) = \lambda_1(\rho(\gamma)) \ .$$

There also exists an embedding[3]

$$J_n : \mathrm{Hit}_n \to \mathrm{Par}_1^h(\Sigma)$$

such that for all $\rho \in Hit_n$ and all $\gamma \in [\Gamma]$,

$$L_{J_n(\rho)}(\gamma) = \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)) = L_\rho^{sw}(\gamma) \ .$$

Remark 3.4.7. The highest weight length spectrum of a Hitchin representation is also given by some reparametrization, but the corresponding map $I'_n : \mathrm{Hit}_n \to \mathbf{P}\mathrm{Par}(\Sigma)$ is not an embedding because a representation ρ has the same highest weight length spectrum as its image by the outer automorphism of $\mathrm{PSL}(n, \mathbb{R})$.

3. The fact that the simple weight entropy of Hitchin representations equals 1 is due to Potrie–Sambarino [158].

These embeddings of Hit_n allow the authors to construct pressure metrics on Hit_n by pulling back the pressure metric on $\mathbf{P}\mathrm{Par}^h(\Sigma)$. By McMullen's theorem, these metrics restrict to the Weil–Petersson metric on the Fuchsian locus. The pressure metric on the tangent space to Hit_n at a Fuchsian point was computed by Labourie and Wentworth [124]. It is not known whether these pressure metrics are Kähler for some complex structure on Hit_n.

Teichmüller space of the stable foliation

Recall that the geodesic flow φ on $T_1\Sigma$ preserves a 2-dimensional foliation $\mathcal{W}^s$, called the *weakly stable foliation*, such that two points in the same leaf remain at bounded distance under the flow for positive time. This foliation is also the weakly stable foliation of any Hölder reparametrization of φ. Our second avatar of a highest Teichmüller space is the space of complex structures on the foliation $\mathcal{W}^s$.

A *foliated complex structure* (resp. *foliated hyperbolic structure*) on $\mathcal{W}^s$ is the data of a complex structure (resp. a hyperbolic metric) along the leaves of $\mathcal{W}^s$, which varies continuously with the leaf. Two such structures are called *homotopic* if one is the image of the other by a homotopy preserving the leaves. The uniformization theorem of Poincaré–Koebe was generalized to 2-dimensional foliations by Candel [40]. In our context, Candel's theorem shows that any foliated complex structure on $\mathcal{W}^s$ admits a unique conformal foliated hyperbolic metric.

Definition 3.4.8. Our second *highest Teichmüller space* is the space $\mathcal{T}(\mathcal{W}^s)$ of homotopy classes of foliated complex structures (or equivalently, of foliated hyperbolic metrics) on $\mathcal{W}^s$.

Teichmüller spaces of foliations were introduced by Sullivan in [183] where he proves that these spaces carry most of the geometry of classical Teichmüller spaces.

Theorem 3.4.9 (Sullivan [183])**.** *The space $\mathcal{T}(\mathcal{W}^s)$ is an infinite dimensional complex Banach manifold, biholomorphic to an open bounded domain in the space of foliated holomorphic quadratic differentials. It carries a complete Finsler metric, analogous to the classical Teichmüller metric.*

Sullivan had the intuition that foliated Teichmüller spaces parametrized certain moduli spaces of dynamical systems. He constructs for instance a correspondence between dilating $\mathcal{C}^1$ self maps of the circle and foliated complex structures on a 2-dimensional lamination associated to the solenoid. In the same spirit, Cawley studies in [42] the "Teichmüller space" of Anosov diffeomorphisms of the torus. In [192], we establish the correspondence between foliated complex structures on $\mathcal{W}^s$ and reparametrizations of the geodesic flow of entropy 1. Denote by $\mathcal{T}^h(\mathcal{W}^s)$ the dense subset of foliated complex structures that are transversally Hölder regular.

Theorem 3.4.10 (Tholozan [192])**.** *There is a continuous map*

$$\mathrm{CF} : \mathcal{T}(\mathcal{W}^s) \to \mathrm{Par}_1(\Sigma)$$

which induces a bijection between $\mathcal{T}^h(\mathcal{W}^s)$ *and* $\mathrm{Par}_1^h(\Sigma)$.

The construction of CF relies on the fact that each leaf of $\mathcal{W}^s$ has a preferred "point at infinity". To a foliated hyperbolic metric, one can associate the flow which follows geodesics going to this point at infinity. This flow is conjugate to a reparametrization of the geodesic flow. Note that this foliated hyperbolic metric similarly defines a "foliated horocyclic flow", which follows on each hyperbolic leaf the horocycles tangent to the preferred point at infinity.

To construct an inverse to CF, one needs, given a reparametrization ψ of the geodesic flow, to recover a horocyclic flow h such that

$$\psi_t \circ h_s \circ \psi_{-t} = h_{e^{-t}s} .$$

This flow is constructed by desintegrating the Bowen–Margulis measure of ψ along the strongly stable leaves. These are well-defined when ψ is Hölder continuous. The fact that the desintegrated Margulis measures are scaled by e^t under ψ_t precisely means that ψ has entropy 1 (see [140]).

The main motivation behind Theorem 3.4.10 is to (re)introduce complex geometry into higher Teichmüller theory. In particular, we will show in a future work that the pressure metric on $\mathrm{Par}_1(\Sigma)$ becomes an infinite dimensional Weil–Petersson metric on $\mathcal{T}(\mathcal{W}^s)$. To be more precise, let us first say a few words about the tangent space to $\mathcal{T}(\mathcal{W}^s)$ at a foliated complex structure J. Like in the classical setting, this tangent space can be seen as the space $H^1(K_J^{-1})$ of $\bar{\partial}$-cohomology classes of foliated Beltrami differentials. Through Theorem 3.4.9 (which is based on the Bers embedding of Teichmüller space), it can also be seen as the space $H^0(K_J^2)$ of foliated holomorphic quadratic differentials.

In the classical setting, the space $H^0(K_J^2)$ is naturally dual to $H^1(K_J^{-1})$, so that their identification is given by a Hermitian metric which is precisely the Weil–Petersson metric. In the foliated setting however, while one can still pair a Beltrami differential with a quadratic differential pointwise, a measure on the whole space $T_1\Sigma$ is needed in order to turn this into a duality.

Fortunately, if we assume that J is transversally Hölder, there is a preferred measure μ_J on $(T_1\Sigma, J)$ called the *harmonic measure*, which corresponds to the unique probability measure of maximal entropy for the reparametrized flow $\mathrm{CF}(J)$. Using this measure we can define and analog of the Weil–Petersson metric of $\mathcal{T}(\mathcal{W}^s)$:

Definition 3.4.11. The Weil–Petersson metric on $\mathcal{T}(\mathcal{W}^s)$ at a foliated hyperbolic metric g_{hyp} is given by

$$h_{WP}(\varphi, \varphi) = \int_{T_1(\Sigma)} \frac{\varphi\bar{\varphi}}{g_{hyp}^2} \mathrm{d}\mu_J ,$$

where φ is a foliated holomorphic quadratic differential.

In a forthcoming work, adapting techniques of McMullen [144] (see also [124]), we will prove that this Weil–Petersson metric corresponds via CF to the pressure metric.

Theorem 3.4.12. *The map* $\mathrm{CF} : \mathcal{T}^h(\mathcal{W}^s) \to \mathrm{Par}_1^h(\Sigma)$ *is* $\mathcal{C}^1$ *and*

$$\mathrm{CF}^* h_{pressure} = \lambda h_{WP} ,$$

for some constant λ *which will be explicitly computed.*

I hope this result is convincing enough of the relevance of studying the space $\mathcal{T}(\mathcal{W}^s)$ as a highest Teichmüller space. This raises two main questions.

Question 3.4.13. *Is the metric* h_{WP} *a Kähler metric ?*

Question 3.4.14. *Does* Hit_n *embed into* $\mathcal{T}(\mathcal{W}^s)$ *as a holomorphic submanifold ?*

A positive answer to both questions would imply the existence of a mapping class group invariant complex structure on Hit_n for which the pressure metric is Kähler.

There exist various proofs that the Weil–Petersson metric on the classical Teichmüller spaces is Kähler, and we strongly hope that one of these proofs can be generalized to the foliated setting. Question 3.4.14, on the other side, seems much harder to address because the embedding of Hit_n into $\mathcal{T}(\mathcal{W}^s)$ factors through the map CF^{-1}, which is hardly computable since it requires finding the Bowen–Margulis measure associated to a reparametrization. We actually have little reason to hope for a positive answer, beyond the following prelimilary result, which follows from Labourie–Wentworth [124]:

Proposition 3.4.15. *The tangent space to* Hit_n *at a Fuchsian point is a complex subspace of the tangent space to* $\mathcal{T}(\mathcal{W}^s)$*.*

Diffeomorphisms of the circle

Our third avatar of a highest Teichmüller space is a certain space of $\mathcal{C}^1$ actions of Γ_g on the circle. It is meant to generalize the identification of $\mathcal{T}_g$ with the component of maximal Euler class in $\mathfrak{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$.

Recall that $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathrm{PU}(1,1)$, which acts by homographies on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. The *Euler class* of a representation ρ

into $\mathrm{PU}(1,1)$ is the Euler class of the flat $\mathbb{S}^1$-bundle associated to ρ, seen as an oriented circle bundle. This definition readily extends to representations into the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of orientation preserving homeomorphisms of the circle.

The Euler class of a representation $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}(\mathbb{S}^1)$ still satisfies the Milnor–Wood inequality $|\mathbf{eu}(\rho)| \leq 2g-2$ [208]. Moreover, representations of maximal Euler class have the following characterization:

Theorem 3.4.16 (Matsumoto [143])**.** *Let $j : \pi_1(\Sigma) \to \mathrm{PU}(1,1)$ be a Fuchsian representation and $\rho : \pi_1(\Sigma) \to \mathrm{Homeo}_+(\mathbb{S}^1)$ a representation of Euler class $2g-2$. Then ρ is semi-conjugate to j.*

Under some regularity hypotheses, this rigidity theorem admits the following stronger generalization:

Theorem 3.4.17 (Ghys [68])**.** *Let ρ be a representation of $\pi_1(\Sigma)$ into the group $\mathrm{Diff}^k(\mathbb{S}^1)$ of diffeomorphisms of class $\mathcal{C}^k$, with $k \geq 3$. Then there exists a Fuchsian representation $j : \pi_1(\Sigma) \to \mathrm{PU}(1,1)$ such that ρ is conjugate to j by a diffeomorphism of class $\mathcal{C}^k$.*

Remark 3.4.18. According to Bertrand Deroin this result is likely to hold in regularity $\mathcal{C}^2$.

Let us denote by $\mathrm{Diff}(\mathbb{S}^1)$ the group of diffeomorphisms of $\mathbb{S}^1$ of class $\mathcal{C}^1$, by $\mathrm{Diff}^h(\mathbb{S}^1)$ the subgroup of diffeomorphisms with Hölder derivatives and by $\mathrm{Diff}^k(\mathbb{S}^1)$ the subgroup of diffeomorphisms of class $\mathcal{C}^k$. If G is a subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ with a topology, define $\mathfrak{X}(\Gamma_g, G)$ to be the largest Hausdorff quotient of $\mathrm{Hom}(\Gamma_g, G)$ under the conjugation action of G, and by $\mathfrak{X}_{\max}(\Gamma_g, G)$ the subset of representations with maximal Euler class. The above rigidity theorems state that $\mathfrak{X}_{\max}(\Sigma_g, \mathrm{Homeo}(\mathbb{S}^1))$ is reduced to a point while $\mathfrak{X}_{\max}(\Gamma_g, \mathrm{Diff}^k(\mathbb{S}^1)) = \mathfrak{X}_{\mathit{Fuchs}}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ for $k \geq 3$.

The $\mathcal{C}^1$ regularity turns out to be much richer: as we will see, the space $\mathfrak{X}(\Gamma_g, \mathrm{Diff}(\mathbb{S}^1))$ "contains" the space of reparametrizations of the geodesic flow. To be more precise, let us introduce a subclass of maximal representations.

Proposition-Definition 3.4.19. *Let ρ be a representation of Γ_g into $\mathrm{Diff}(\mathbb{S}^1)$. The following properties are equivalent:*

- *ρ has extremal Euler class, acts minimally on $\mathbb{S}^1$ and is expanding, i.e. for all $x \in \mathbb{S}^1$, there exists $\gamma \in \Gamma_g$ such that*

$$|\rho(\gamma)'(x)| > 1 \ ,$$

- *ρ is Hölder conjugate to some (hence any) Fuchsian representation.*

We call such a representation an Anosov representation into $\mathrm{Diff}(\mathbb{S}^1)$.

Let $\mathfrak{X}_{an}(\Sigma, \mathrm{Diff}(\mathbb{S}^1))$ denote the open set of equivalence classes of Anosov representations of $\pi_1(\Sigma)$ into $\mathrm{Diff}(\mathbb{S}^1)$, and by $\mathfrak{X}_{an}(\Sigma, \mathrm{Diff}^h(\mathbb{S}^1))$ the dense subset of Anosov actions with Hölder derivatives.

Theorem 3.4.20. *There is a continuous surjective map*

$$\mathrm{DF} : \mathfrak{X}_{an}(\Sigma, \mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Par}_1(\Sigma)$$

which restricts to a bijection from $\mathfrak{X}_{an}(\Sigma, \mathrm{Diff}^h(\mathbb{S}^1))$ *to* $\mathrm{Par}_1^h(\Sigma)$.

Given a representation $\rho : \pi_1(\Sigma) \to \mathrm{Diff}(\mathbb{S}^1)$, define the *period map* of ρ by

$$\begin{array}{rccl} L_\rho : & [\Gamma] & \to & \mathbb{R}_+ \\ & \gamma & \mapsto & -\log\left(\rho(\gamma)'(\gamma_+)\right) \ , \end{array}$$

where γ_+ denotes the attracting fixed point of $\rho(\gamma)$ on $\mathbb{S}^1$. Then the map DF preserves periods, i.e. $L_{\mathrm{DF}(\rho)} = L_\rho$. This characterizes DF.

While the construction of the map DF is rather explicit, its converse relies once again on the existence of Margulis measures. More precisely, the desintegration of the Bowen–Margulis measure on strongly unstable leaves of a reparametrization ψ define a $\mathcal{C}^1$ structure [4] on the space of stable leaves $\partial_\infty \Gamma_g$, which one can see as a $\mathcal{C}^1$ action of $\pi_1(\Sigma)$ on the circle which is topologically conjugate to a Fuchsian action.

The surjectivity of DF in general requires to carefully check what persists of Margulis construction in the absence of the Hölder regularity hypotheses. It is unclear whether DF is injective. We have reasons to believe that there exist distinct Anosov actions ρ_1 and ρ_2 on $\mathbb{S}^1$ which are not $\mathcal{C}^1$-conjugate but have the same periods. However, in this situation, it is plausible that ρ_2 is a limit of conjugates of ρ_1, so that the two would be identified in the Hausdorff quotient $\mathfrak{X}_{an}(\Sigma, \mathrm{Diff}(\mathbb{S}^1))$.

An important component of Ghys's rigidity theorem (Theorem 3.4.17) is that every representation of $\pi_1(\Sigma)$ into $\mathrm{Diff}^k(\mathbb{S}^1)$, $k \geq 3$ with maximal Euler class is Anosov in the sense of Definition 3.4.19. In particular such an action is minimal [67]. An interesting question is whether this result extend to $\mathcal{C}^1$ regularity. We will answer it negatively in a forthcoming work with Ghazouani and Dal'bo:

Theorem 3.4.21 (Dal'bo–Ghazouani–Tholozan)**.** *There exist*

- *representations* $\rho : \pi_1(\Sigma) \to \mathrm{Diff}(\mathbb{S}^1)$ *of maximal Euler class which act minimally on* $\mathbb{S}^1$ *but are not dialating,*
- *representations* $\rho : \pi_1(\Sigma) \to \mathrm{Diff}(\mathbb{S}^1)$ *of maximal Euler class with an exceptional minimal set of positive measure,*
- *representations* $\rho : \pi_1(\Sigma) \to \mathrm{Diff}(\mathbb{S}^1)$ *of maximal Euler class with an exceptional minimal set of measure* 0.

In order to construct such examples, one wants to understand limits of Anosov representations into $\mathrm{Diff}(\mathbb{S}^1)$. Our analogy with Anosov representations into linear groups turn out to be fruitful here: we obtain these

4. Alternately, one can get this $\mathcal{C}^1$ structure via a Patersson–Sullivan construction.

examples using Patterson–Sullivan measures associated to discrete and faithful representations of $\pi_1(\Sigma)$ into $\mathrm{PSL}(2,\mathbb{C})$ that are limits of quasi-Fuchsian representations.

We plan to investigate further these examples, and hope to obtain a clear description of all maximal representations into $\mathrm{Diff}(\mathbb{S}^1)$.

3.4.2 Branched hyperbolic structures with prescribed holonomy

Recall that Goldman conjectured that every faithful representation $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ of Euler class $k \geq 1$ should be the holonomy of a hyperbolic structure on Σ_g with $2g-2-k$ branched points.

There are at least three tools that might help tackle this conjecture:

— **Geometric surgery:** The perhaps most natural approach would consist in cutting Σ_g into smaller pieces in restriction to which ρ is the holonomy of a branched hyperbolic structure (with a controlled behaviour at the boundary components), and then glue these pieces together via some "hyperbolic surgery". One can invent many such surgeries, but the difficult part in this approach is to find the appropriate decomposition of Σ_g.

— **Apply mapping class group transformations:** In order to implement the first approach, one might want to move in the space of "decompositions" of Σ_g by applying a well-chosen mapping class group transormation. Equivalently, one might try to replace ρ by its image under a mapping group element. Good recurrence properties of this action might help get close enough to a representation that we understand well, and conclude using the fact that holonomies of branched hyperbolic structures form an open $\mathrm{MCG}(\Sigma_g)$-invariant set. In particular, Golman's ergodicity conjecture (Conjecture 3.1.11) would imply that holonomies of branched hyperbolic structures form a subset of full measure in $\mathfrak{X}_k(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$, $k \geq 1$.

— **Minimizing the energy functional:** Finding a branched hyperbolic structure with holonomy ρ is equivalent to finding a complex structure J on Σ_g, such that the unique ρ-equivariant harmonic map $f_{J,\rho}$ is holomorphic.This is a special case of the problem of finding a branched minimal immersion. As explained in Section 3.2.1, it boils down to proving that the energy functional $\mathbf{E}_\rho$ admits a critical point. Unfortunately, this energy functional is not proper unless ρ is Fuchsian. In fact, its set of minima is a (possibly empty) holomorphic submanifold of dimension $2g-2-\mathbf{eu}(\rho)$.

With Bertrand Deroin, we have been working on a combination of these three approaches. At the price of a lot of technicality, we believe we managed to prove the following theorem:

Theorem 3.4.22 (Deroin–T.)**.** *Let $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ be a representation of Euler class $2g-3$. Then ρ is the holonomy of a hyperbolic structure on Σ_g with one branched point.*

3.4.3 Bounded mapping class group orbits

While studying branched hyperbolic metrics on surfaces, Bertrand Deroin and I discovered somehow serendipitously the existence of our "supra-maximal" representations of the fundamental group of a punctured sphere into $\mathrm{PSL}(2,\mathbb{R})$. Poundering on those examples lead us to question more generally the properties of bounded mapping class group orbits in character varieties. This opens a broad research program which is still at its early stages. We conclude this chapter by outlining this program.

In this section, we consider more generally representations of a surface group with punctures. A first motivation for that is to have more examples (including those constructed in [56] and [193]) in hope that understanding their common features will give us a better intuition of the general underlying phenomena.A second motivation is that the class of representations with bounded mapping class group orbits is stable under restriction to a subsurface, so that one can hope to understand them via a "cutting/gluing" operation which already played an important role in [56].

Let thus $\Sigma_{g,n}$ be an oriented surface of genus g with n punctures, $\Gamma_{g,n}$ its fundamental group, and G a semisimple Lie group.

Definition 3.4.23. A representation $\rho : \pi_1(\Sigma) \to G$ has *bounded mapping class group orbit* if

$$\{\sigma \cdot [\rho], \sigma \in \mathrm{MCG}_{g,n}\}$$

is contained in a compact subset of $\mathfrak{X}(\Sigma_{g,n}, G)$.

Diverse examples

Let us start by giving various examples of such bounded orbits.

Representations into compact groups: The first and somehow trivial example of a representation with bounded mapping class group orbit is a representation with values into a compact subgroup of G. In fact, when ρ takes values into a compact group, then the family

$$\{\rho \circ \sigma \ , \sigma \in \mathrm{Aut}(\Gamma_{g,n})\}$$

is bounded in $\mathrm{Hom}(\Gamma_{g,n}, G)$.

In contrast, the examples we further present do not take values into a compact group and, while their $\mathrm{Aut}(\Gamma_{g,n})$-orbit is not bounded in $\mathrm{Hom}(\Gamma_{g,n}, G)$, it becomes so after moding out by conjugation.

Fixed points and finite orbits: A particular type of bounded mapping class group orbit consists of fixed points under the mapping class group action. Though their existence (beyond the trivial representation) is far from obvious, such fixed points are known to exist and arise from linear representations of mapping class groups.

To be more precise, recall the existence of the *Birman exact sequence*:

$$\mathbf{1} \to \pi_1(\Sigma_{g,n}) \to \mathrm{MCG}_{g,n+1} \to \mathrm{MCG}_{g,n} \to \mathbf{1}\ .$$

The image of $\pi_1(\Sigma_{g,n})$ in $\mathrm{MCG}_{g,n+1}$ consists of mapping class group elements which move the last puncture around $\Sigma_{g,n}$, and the morphism $\mathrm{MCG}_{g,n+1} \to \mathrm{MCG}_{g,n}$ "forgets" the last puncture. The conjugation action of $\mathrm{MCG}_{g,n+1}$ gives an embedding $\mathrm{MCG}_{g,n+1} \hookrightarrow \mathrm{Aut}(\Gamma_{g,n})$ such that $\Gamma_{g,n}$ acts on itself by inner automorphisms.

Let ρ be a representation of $\mathrm{MCG}_{g,n+1}$ into some linear group G. For all $\gamma \in \Gamma_{g,n}$ and all $\sigma \in \mathrm{MCG}_{g,n+1} \subset \mathrm{Aut}(\Gamma_{g,n})$, we have

$$\rho(\sigma \cdot \gamma) = \rho(\sigma)\rho(\gamma)\rho(\sigma)^{-1}$$

by the Birman exact sequence. It follows that the conjugacy class of $\rho_{|\Gamma_{g,n}}$ in $\mathfrak{X}(\Gamma_{g,n}, G)$ is fixed by the action of $\mathrm{MCG}_{g,n}$.

Conversely, let ρ be a representation of $\Gamma_{g,n}$ into G whose conjugacy class is a fixed point of the action of $\mathrm{MCG}_{g,n}$. Assume moreover that the image of ρ has trivial centralizer. Then for every $\sigma \in \mathrm{MCG}_{g,n+1} \subset \mathrm{Aut}(\Gamma_{g,n})$, the representation $\rho \circ \sigma^{-1}$ is conjugate to ρ by a unique element $\widehat{\rho}(\sigma) \in G$. One easily verifies that $\widehat{\rho}$ is a representation of $\mathrm{MCG}_{g,n+1}$ which restricts to ρ on $\Gamma_{g,n}$.

This shows that finding fixed points of the mapping class group action on $\mathfrak{X}(\Sigma_{g,n}, G)$ is essentially equivalent to finding representations of $\mathrm{MCG}_{g,n+1}$ into G. While one might be convinced from this remark that fixed points of the mapping class group are abundant, I tend to see it as a justification why mapping class group representations are hard to construct.

In any case, an interesting family of linear representations of $\mathrm{MCG}_{g,n+1}$ has been constructed by Reshetikhin and Turayev using Topological Quantum Field Theory [164]. These representations typically take values into some pseudo-Hermitian Lie group $\mathrm{SU}(r,s)$. Koberda and Santharubane recently proved that the restrictions of (some of) those representations to $\Gamma_{g,n}$ are irreducible (in particular, they do not take value into a compact subgroup) [113]. This provides, for every pair (g,n), an infinite family of fixed points of $\mathrm{MCG}_{g,n}$ parametrized by a root of unity q. However, the rank of these representations grows quickly with g, n, and q. Thus, TQFT, while providing examples, is very far from giving a conjectural picture of the set of fixed points of the mapping class group action in a given character variety.

More generally, finite orbits of the action of $\mathrm{MCG}_{g,n}$ on $\mathfrak{X}(\Gamma_{g,n}, G)$ correspond to representations of finite index subgroups of $\mathrm{MCG}_{g,n+1}$. Finite orbits of $\mathrm{MCG}_{0,4}$ acting on $\mathfrak{X}(\Gamma_{0,4}, \mathrm{PSL}(2,\mathbb{C}))$ are realted to algebraic solutions of Painlevé VI equations and have been classified by Lisovyy and Tykhyy [130]. Little is known about them in higher rank or genus.

Bounded relative components for punctured spheres: While finite mapping class group orbits are (at least conjecturally) scarce, Theorems 3.3.1 and 3.3.3 show that bounded mapping class group orbits might be much more abundant, and do in fact cover open subsets of some character varieties in genus 0.

Katz's middle convolution: The *middle convolution* operation, introduced by Katz in [99], provides a way to construct new bounded mapping class group orbits in character varieties of punctured spheres. Introducing it properly would bring us too far from our topic, so we will only briefly sketch how it enters in the picture.

This operation transforms a local system on the punctured sphere $\Sigma_{0,n}$ into another local system of different rank, by a process that ressembles a Fourier–Mukai transform. What is remarkable for our purpose is that it provides $\mathrm{MCG}_{0,n}$-equivariant isomorphisms between certain strata of relative character varieties in $\mathrm{SL}(d,\mathbb{C})$ for different values of d.

Let now ρ be a representation of $\Gamma_{0,n}$ into a compact subgroup of $\mathrm{SL}(d,\mathbb{C})$. By mapping class group equivariance, applying the middle convolution to ρ gives a representation into some $\mathrm{SL}(d',\mathbb{C})$ with bounded mapping class group orbit. There are examples where the middle convolution maps representations into a compact group to representations with unbounded images, yielding non trivial bounded orbits.

Katz introduced the middle convolution in order to classify "rigid representations", i.e. relative strata which are reduced to a point. Note in particular that these are fixed points of the mapping class group action. Katz proved that they can all be obtained from a rank 1 local system by applying iteratively the middle convolution. One could more generally hope that bounded mapping class group orbits in character varieties of punctured spheres can all be obtained by applying the middle convolution to representations into compact groups.

Though the middle convolution opertation does not generalize well to surfaces we genus, Jérémy Toulisse and I may have found a similar construction which could produce examples of bounded orbits in character varieties of closed surfaces.

Conjectural picture

In this last section, we give a conjectural picture of bounded mapping class group orbits based on the previous examples, and support it with preliminary results obtained with Bertrand Deroin. These results are far from taking written form and should be taken with caution.

Let us call a representation $\rho : \Sigma_{g,n} \to G \subset \mathrm{SL}(d, \mathbb{C})$ *cusp preserving* if the image of every peripheral curve has eigenvalues of module 1. This is equivalent to the existence of a ρ-equivariant map of finite energy.

Proposition 3.4.24. *Let ρ be a linear representation of $\Gamma_{g,n}$ into G. The following are equivalent:*

(*i*) *ρ is cusp preserving,*

(*ii*) *for every complex structure J on $\Sigma_{g,n}$, there exists a ρ-equivariant map from $(\widetilde{\Sigma}_{g,n}, J)$ to the symmetric space G/K of finite energy.*

In the remainder of this section we only consider cusp-preserving representations. The following lemma gives a motivation for that:

Lemma 3.4.25. *Let ρ be a cusp preserving representation of $\Gamma_{g,n}$ into* $\mathrm{SL}(d, \mathbb{C})$ *with bounded mapping class group orbit. Then the image of any simple closed curve γ has all its eigenvalues of module* 1.

In other words, the class of cusp preserving representations with bounded mapping class group orbit is stable under restriction to a subsurface.

Remark 3.4.26. There exist representations of $\Gamma_{0,n}$ into $\mathrm{PSL}(2, \mathbb{C})$ with finite mapping class group orbit which are not cusp preserving.

To formulate our main conjecture, let us recall the definition of a *complex variation of Hodge structure*, as defined in [178].

Definition 3.4.27. A (parabolic) Higgs bundle $(\mathcal{E}, \Theta)$ on a (punctured) Riemann surface $(\Sigma_{g,n}, J)$ is a *complex variation of Hodge structure* of weight k if $\mathcal{E}$ admits a holomophic splitting as

$$\mathcal{E}_0 \oplus \ldots \oplus \mathcal{E}_k$$

such that Θ maps $\mathcal{E}_i$ into $\mathcal{K}_J \otimes \mathcal{E}_{i+1}$.

Remark 3.4.28. This terminology comes from Hodge theory: the cohomology of a holomorphic family of projective varieties over a complex base B defines a local system over B. The associated Higgs bundle over B is a variation of Hodge structure in this sense, the splitting of which is given by the varying Hodge decomposition of the cohomology of the fiber. The required property of Θ follows from the *Griffith transversality.*

Variations of Hodge structure are the fixed points of the natural action of $\mathbb{C}^*$ on the moduli space of Higgs bundles given by $\lambda \cdot (\mathcal{E}, \Theta) = (\mathcal{E}, \lambda\Theta)$, and are thus the critical points of Hitchin's energy functional $\rho \mapsto \mathbf{E}(J, \rho)$.

We can now state our general conjecture for bounded mapping class group orbits:

Conjecture 3.4.29. *Let ρ be a cusp-preserving representation of $\Gamma_{g,n}$ into $\mathrm{SL}(d, \mathbb{C})$. Then the following are equivalent:*
(*i*) *The $\mathrm{MCG}_{g,n}$-orbit of $[\rho]$ is bounded in $\mathfrak{X}(\Sigma_{g,n}, \mathrm{SL}(d, \mathbb{C}))$,*
(*ii*) *The energy functional $\mathbf{E}_\rho$ on $\mathcal{T}(\Sigma_{g,n})$ is constant,*
(*iii*) *For all complex structures J on $\Sigma_{g,n}$, the Higgs bundle over $(\Sigma_{g,n}, J)$ associated to ρ is a variation of Hodge structure.*

The conjecture in examples: This conjecture is first motivated by the examples discussed in Section 3.4.3.

— If ρ takes values in a compact group, then the energy functional $\mathbf{E}_\rho$ vanishes identically, and any Higgs bundle associated to ρ has vanishing Higgs field. It is thus a variation of Hodge structure of weight 0.

— The representations in compact relative components of character varieties that we construct in [56] and [193] also have constant energy functional. For every complex structure on $\Sigma_{0,n}$, the associated ρ-equivariant harmonic map to G/K is holomorphic, which transcribes into the associated Higgs bundle being a variation of Hodge structure of weight 1.

— Though this may require some clarification, it seems to follow from Szabo's work [184] that the Higgs bundle counterpart of Katz's middle convolution operation transforms a variation of Hodge structure of weight k into a variation of Hodge structure of weight $k + 1$. Hence the bounded orbits obtained by applying the middle convolution to representations with values in compact groups should be variations of Hodge structures.

Preliminary results: Conjecture 3.4.29 is also supported by some preliminary results that we obtained with Bertrand Deroin. A first result is that a bounded mapping class group orbit has bounded energy functional:

Lemma 3.4.30 (Deroin–Tholozan)**.** *Let ρ be a cusp-preserving representation of $\Gamma_{g,n}$ into $\mathrm{SL}(d, \mathbb{C})$. Then the following are equivalent:*
(*i*) *The $\mathrm{MCG}_{g,n}$-orbit of $[\rho]$ is bounded in $\mathfrak{X}(\Sigma_{g,n}, \mathrm{SL}(d, \mathbb{C}))$,*
(*ii*) *The energy functional $\mathbf{E}_\rho$ on $\mathcal{T}_{g,n}$ is bounded.*

On the other side, the energy functional $\mathbf{E}_\rho$ is known to be pluri-subharmonic by a theorem of Toledo [198]. Using the fact that a pluri-subharmonic function that achieves its maximum is constant, we obtain the following

Theorem 3.4.31 (Deroin–Tholozan)**.** *ρ be a cusp-preserving representation of $\Gamma_{g,n}$ into $\mathrm{SL}(d,\mathbb{C})$ with bounded mapping class group orbit. Then there exists $[\rho']$ in the closure of the orbit of $[\rho]$ such that $\mathbf{E}_{\rho'}$ is constant. In particular, if the action of $\mathrm{MCG}_{g,n}$ on $\overline{\mathrm{MCG}_{g,n}\cdot[\rho]}$ is minimal, then $\mathbf{E}_\rho$ is constant.*

The next step is to study representations with constant energy functional. These satisfy the equality in Toledo's pluri-subharmonicity theorem, which should imply some rigidity property. To make this more explicit, we found a Gauge theoretical proof of Toledo's subharmonicity adapted to the Higgs bundle point of view. Understanding the equality case gives us that Higgs bundles associated representations with constant energy functional are highly critical points of the *Hitchin fibration.*

For a complex structure J on Σ_g, Let $\mathcal{M}_{Higgs}(J,d)$ denote the moduli space of semistable $\mathrm{SL}(d,\mathbb{C})$-Higgs bundles on (Σ_g,J). Define

$$\begin{array}{rccc} \chi^i_{J,d}: & M_{Higgs}(J,d) & \to & H^0(K_J^i) \\ & (E,\Theta) & \mapsto & \mathrm{Tr}(\Theta^i) \end{array}.$$

The *Hitchin fibration* [88] is the map $(\chi^2,\ldots,\chi^d)$. It is a generic submersion whose fibers are half dimensional abelian varieties.

Theorem 3.4.32 (Deroin–Tholozan)**.** *Let $\rho:\Gamma_{g,0}\to\mathrm{SL}(d,\mathbb{C})$ be a representation and J be a critical point of $\mathbf{E}_\rho$ such that the Hessian of $\mathbf{E}_\rho$ at J vanishes. Then the differential of $\chi^2_{J,d}$ vanishes at the Higgs bundle corresponding to ρ.*

Though we haven't been able yet to give a clear interpretation of the above property, it implies for instance that the Higgs bundle (E,Θ) associated to (J,ρ) does not have a regular spectral curve. This is consistent with the hope that Θ should be nipotent as it is the case in variations of Hodge structures.

Higgs bundles over the moduli space: In the particular case of fixed points of the Mapping class group action, Conjecture 3.4.29 is related to the question of whether linear representations of mapping class groups are rigid. Indeed, we have the following proposition, adapted from a classical argument communicated to me by Jérémy Daniel:

Proposition 3.4.33. *Assume that there exists a complex structure J on Σ_g such that every fixed point of the action of MCG_g on $\mathfrak{X}(\Sigma_g,\mathrm{SL}(d,\mathbb{C}))$ is associated to a variation of Hodge structure on (Σ_g,J). Then the fixed locus of MCG_g in $\mathfrak{X}(\Sigma_g,\mathrm{SL}(d,\mathbb{C}))$ is finite.*

Proof. The character variety $\mathfrak{X}(\Sigma_g, \mathrm{SL}(d,\mathbb{C}))$ is defined over $\mathbb{R}$. The locus of fixed points of MCG_g is a complex subvariety of $\mathfrak{X}(\Sigma_g, \mathrm{SL}(d,\mathbb{C}))$. If it has positive dimension, it must contain points which are not real. On the other side, every variation of Hodge structure gives rise to a representation into some $\mathrm{SU}(r,s)$, which is thus a real point of $\mathfrak{X}(\Sigma_g, \mathrm{SL}(d,\mathbb{C}))$. Hence, if all fixed points are of this form, then the fixed locus is zero dimensional. $\square$

Let now ρ be a representation of $\mathrm{MCG}_{g,1}$ into $\mathrm{SL}(d,\mathbb{C})$. If the restriction of ρ to Γ_g is irreducible, then $\rho_{|\Gamma_g}$ completely characterizes ρ. In particular, if $\rho_{|\Gamma_g}$ is an isolated fixed point of $\mathfrak{X}(\Sigma_g, \mathrm{SL}(d,\mathbb{C}))$, then the representation ρ is rigid. Using the result of Santharoubane and Koberda, we deduce for instance:

Proposition 3.4.34. *If Conjecture 3.4.29 holds, then the Reshetykhin–Turayev representations of* $\mathrm{MCG}_{g,1}$ *studied in [113] are rigid.*

With a bit more work, one can hope to prove that Conjecture 3.4.29 implies the rigidity of all mapping class group representations.

List of personal publications

Here is a list of my publications and prepublications.

Publications

- Nicolas Tholozan, “Sur la complétude de certaines variétés pseudo-riemanniennes localement homogènes.” *Annales de l’institut Fourier* **65**(5), 2015, p. 1921-1952.

- Bertrand Deroin and Nicolas Tholozan, “Dominating surface group representations by Fuchsian ones.” *International Mathematics Research Notices* **2016** (13), 2016, p. 4145-4166.

- Nicolas Tholozan, “Dominating surface group representations and deforming closed anti-de Sitter 3-manifolds.” *Geometry and Topology* **21**, 2017, p. 193-214.

- Nicolas Tholozan, “Entropy of Hilbert metrics and length spectrum of Hitchin representations in $\mathrm{PSL}(3,\mathbb{R})$.” *Duke Mathematical Journal* **166** (7), 2017, p. 1377-1403.

- Nicolas Tholozan, “The Volume of complete anti-de Sitter 3-manifolds.” *Journal of Lie theory* **28** (3), 2018, p. 619-642.

- Bertrand Deroin and Nicolas Tholozan, “Supra-maximal representations from fundamental groups of punctured spheres into $\mathrm{PSL}(2,\mathbb{R})$.” *Annales Scientifiques de l’École Normale Supérieure* **52** (5), 2019, p. 1305-1329.

- Brian Collier, Nicolas Tholozan and Jérémy Toulisse, “The Geometry of maximal representations of surface groups into $\mathrm{SO}(2,n)$.” *Duke Mathematical Journal* **168** (15), 2019, p. 2873-2949.

- Nicolas Tholozan and Jérémy Toulisse, “Compact components in rel-

ative character varieties of punctured spheres." To appear in *Épijournal de géométrie algébrique.*

Prepublications

— Nicolas Tholozan, "Volume and non-existence of compact Clifford-Klein forms." arXiv:1511.09448.

— Olivier Glorieux, Daniel Monclair and Nicolas Tholozan, "Hausdorff dimension of limit sets for projective Anosov representations." arXiv:1902.01844.

Works in preparation

I only cite here the works in preparation that are mentioned in this memoir.

— Jean-Marc Schlenker, Daniel Monclair and Nicolas Tholozan, "Gromov–Thurston manifolds and anti-de Sitter geometry." In preparation.

— Nicolas Tholozan, "Teichmüller geometry in the Highest Teichmüller space." Lecture notes from an FRG Lecture Series at University of Michigan. Preliminary version available at `https://www.math.ens.fr/~tholozan/Annexes/CocyclesReparametrizations2.pdf`

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