Math. Ann. 292, 685–710 (1992)

Mathematische
Annalen


# Symplectic topology as the geometry of generating functions

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Received November 5, 1990; in revised form July 7, 1991

## Introduction

In the last years important progress in symplectic topology have been achieved using either of the following approaches:

The Gromov-Floer approach studies pseudo holomorphic spheres in symplectic manifolds to infer results on the symplectic topology of the manifold. In particular, results on the geometry of Lagrange submanifolds, existence of fixed points of symplectomorphisms isotopic to the identity, have thus been obtained for which we refer to [G1, Fl1, Fl2]. This approach is essentially the only one available when working in arbitrary symplectic manifolds. In a certain sense we might say that this approach considers *symplectic topology as the study of pseudo-holomorphic curves in almost complex manifolds.*

The other approach, which is the one we are interested in here, could be called the Conley-Zehnder approach. It is mainly concerned with periodic orbits of Hamiltonian systems. This can be a very efficient way of dealing with symplectic topology questions, at least in $\mathbb{R}^{2n}$, as has been sufficiently demonstrated for instance by Ekeland and Hofer in [E-H2, E-H3, V3] and the author [V2]. The method is based on studying the topology of the action functional, this is what we now explain. Let $H(t,x)$ be a time dependent Hamiltonian on $\mathbb{R}^{2n}$, we search 1-periodic solutions of $\dot{x}=J\nabla H(t,x)=X_H(t,x)$ where $dH(t,x)\xi=\omega(X_H(t,x),\xi) = \langle J\nabla H(t,x),\xi\rangle$ (if we identify $\mathbb{R}^{2n}$ to $\mathbb{C}^n$, then $J$ is the matrix of multiplication by $i$). Such periodic solutions can be obtained as critical points of the action functional:

$$A_H(x)=\int_{S^1}\langle Jx,\dot{x}\rangle - H(t,x)dt.$$

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The usual way to get these critical points is by taking minmax on some classes of sets invariant by the flow. While this would be ackward to describe in this introduction, we point out that it becomes the usual minmax theory if we replace $A_H$ by a finite dimensional reduction $A_H^N$. The capacities of $H$ are then the critical value associated to a minmax on certain cohomology classes (cf. [V3]). It is thus fundamental to compute the topology of the sublevel sets $\{x \mid A_H^N(x) \leq \lambda\}$. Thus we can say that this approach considers *symplectic topology as the topology of the action functional.*

Our approach is based on the remark that the action functional is nothing else than a special generating function (for the definition of this we refer to Sect. 1). This was already exploited in [V1] to show that all the indices defined for periodic solutions of Hamiltonian systems are, up to a constant, all equal, and that this indices have a very geometric definition. The natural generalization of such a result is to show that the sublevel sets of a generating function are, after a suitable suspension, diffeomorphic (provided we restrict ourselves to generating functions with quadratic phase cf. Sect. 1). Thus we can get rid of action functionals, and work with any generating function of the time one flow of the Hamiltonian, which is a much more flexible tool. The ideas developed here originated from a question that Yasha Eliashberg asked to Helmut Hofer and myself. Hofer's answer is contained in [H]. The present paper is an expanded version of our own answer which is Corollary 4.8. In the course of the proof which was based on the usual study of the "action functional topology" it appeared that chosing between the "broken geodesic" reduction of Chaperon-Laudenbach-Sikorav, and the Lyapounov-Schmidt reduction of Conley-Zehnder, meant having to chose between two equally desirable properties, $C^0$ continuity of the capacities (Proposition 4.6), and the "triangle inequality" (Proposition 4.8). The advantage of our approach is that we get both properties, moreover most of the proofs are then obvious. The paper is organized as follows.

In Sect. 1 we define generating functions, and prove that they are essentially unique.

Section 2 associates to every Lagrange submanifold $L$ of $T^*B$ and cohomology class $u$ in $H^*(B)$ a real number $c(u, L)$. In this section and the next one we prove basic properties of this numbers.

In Sect. 4 we associate to every compact supported symplectomorphism $\psi$ of $\mathbb{R}^{2n}$ isotopic to the identity (in the set of compact supported symplectomorphisms) a Lagrange submanifold $\Gamma_\psi$ of $T^*S^{2n}$ which is the compactification of its graph. The construction of Sect. 3 then yields two numbers that we shall denote by $c_+(\psi)$ and $c_-(\psi)$.

This allows us to define positive symplectomorphisms [by the condition $c_-(\psi)=0$] and to prove several remarkable properties of these maps.

We also prove that a compact supported symplectomorphism has infinitely many periodic points in the interior of its support.

Section 5 is devoted to studying the behaviour of our invariants by symplectic reduction. The treatment is in no way complete and is only pursued as needed to give a simple proof of the "camel problem", Sect. 6 deals with a property of "simple hypersurfaces".

## 1 Some properties of generating functions with quadratic phase

We first recall some definitions about generating functions.

Let $E \xrightarrow{\pi} B$ be a vector bundle over the manifold $B$, $S: E \to \mathbb{R}$ a $C^2$ function. We shall denote by $\frac{\partial S}{\partial \xi}(e)$ the fiber derivative of $S$ at $e \in E$. We assume the map $\frac{\partial S}{\partial \xi}: E \to E^*$ to be transverse to 0, we may then define the manifold

$$\Sigma_S = \left\{ e \in E \,\middle/\, \frac{\partial S}{\partial \xi}(e) = 0 \right\},$$

and the map

$$i_S: \Sigma_S \to T^*B,$$
$$e \to (\pi(e), \partial S(e)).$$

Note that because $\frac{\partial S}{\partial \xi}(e) = 0$ for $e$ in $\Sigma_S$, we can identify $dS(e)$ to an element of $T^*_{\pi(e)}B$. To simplify the notations we shall often take "abstract local coordinates" $(b, \xi)$ on $E$, we then write $i_S(b, \xi) = \left( b, \frac{\partial S}{\partial b}(b, \xi) \right)$.

It is clear that $i_S$ is a Lagrange immersion. More generally, given a Lagrange immersion $\varphi: L \to T^*B$, we shall say that $S$ is a generating function for $\varphi$ if there is a diffeomorphism $h: L \to \Sigma_S$ such that $i_S \circ h = \varphi$.

Without further assumptions, the question of existence and uniqueness of generating functions for a given Lagrange immersion is completely solved in terms of algebraic topology invariants of the immersion (cf. [Gi, La]), however we shall consider the following class of generating functions.

**Definition 1.1.** A generating function $S$ is said to be *quadratic at infinity* if $S(b, \xi) = q_\infty(b, \xi)$ for $|\xi|$ large enough, where $q_\infty$ is a nondegenerate quadratic form on each fiber.

We shall abbreviate "generating function quadratic at infinity" into "g.f.q.i.". It seems that algebraic topological methods cannot decide whether a Lagrange immersion has a g.f.q.i. or not.

We refer to [Si1, Si2] for examples and counterexamples, and quote the main result from there

**Proposition 1.2.** *The property of having a g.f.q.i. is invariant by Hamiltonian isotopy.*

*Note.* By Hamiltonian isotopy we mean the time one flow of a time dependent Hamiltonian vector field.

The main new result of this section will be a uniqueness result for g.f.q.i. Before we state it, we need to define equivalence of g.f.q.i. (cf. [Hö, We]).

**Definition 1.3.** Let $S_1: E_1 \to \mathbb{R}$, $S_2: E_2 \to \mathbb{R}$ be two g.f.q.i. We shall say that $S_1$ and $S_2$ are *equivalent* if there is a fiber preserving diffeomorphism $\Phi: E_1 \to E_2$, such that $S_2 \circ \Phi = S_1 + \mathrm{Cst}$.

We will usually write $\Phi(b, \xi) = (b, \varphi(v, \xi))$ so that $S_2(b, \varphi(b, \xi)) = S_1(b, \xi) + C$. Clearly $S_1$ and $S_2$ generate the same Lagrange immersion.

**Definition 1.4.** Let $S_1:E_1\to\mathbb{R}$ be a g.f.q.i. and $q_2:E_2\to\mathbb{R}$ a nondegenerate quadratic form on the fibers. Then $S_2:E_1\oplus E_2\to\mathbb{R}$ given by

$$S_2(b,\xi_1,\xi_2)=S_1(b,\xi_1)+q_2(b,\xi_2)$$

will be called a *stabilization* of $S_1$.

Once again, $S_1$ and $S_2$ generate the same Lagrange immersion. Our main result in this section can be stated as

**Proposition 1.5.** *Let $S_1, S_2$ be two g.f.q.i. generating the same Lagrange embedding i of B into $T^*B$. Assume that $i(B)=\varphi_1(O_B)$, where $\varphi_t$ is a Hamiltonian isotopy of $T^*L$. Then after stabilization, $S_1$ is equivalent to $S_2$.*

The proof will take up the end of this section. It is based on the following lemma

**Lemma 1.6.** *If $S_1, S_2$ are g.f.q.i. for the zero section of $T^*B$, then after stabilization $S_1$ and $S_2$ are equivalent.*

*Proof.* We may of course assume $S_1$ to be a quadratic form in the fiber, that is

$$S_1(b,\xi^+,\xi^-)=|\xi^+|^2-|\xi^-|^2,$$

so that $\Sigma_1=B\times\{0\}$.

Now Theorem 4.1.10 of [La], (or [Ce, Proposition 4, p. 168]), states that there is a fiber preserving diffeomorphism $\varphi:U_1\to U_2$, where $U_1, U_2$ are neighbourhoods of $\Sigma_1,\Sigma_2$, such that

$$S_2\circ\varphi=S_1+C$$

(provided $S_1$ and $S_2$ have been suitably stabilized).

*Remarks.* In [La], only generating functions on trivial bundles are considered, and stabilization is only done with a quadratic form independent on the base variable. Then the quoted theorem of [La] actually states that the set of equivalence classes of germs of generating functions under fiber preserving diffeomorphisms is isomorphic to an affine space on the vector space $[X, BO]$ the space of stable vector bundles over $X$. Then in 4.4.3, (loc. cit.) the action of $[X, BO]$ is explicited; if $\alpha\in[X,BO]$ is represented by the vector bundle $A$, and $Q_A$ is a quadratic form on each fiber of trivial vector bundle with negative bundle $A$, then our action is given by $S\to S+Q_A$. Thus Latour's theorem really implies the local equivalence of generating functions with our definition of equivalence.

We now show how $\varphi$ can be extended to a fiber preserving diffeomorphism $\Phi:E_1\to E_2$, such that $S_2\circ\Phi=S_1+C$.

Consider on $E_1$ the metric induced by the Euclidean metric $|\xi^+|^2+|\xi^-|^2$ on each fiber, and endow $E_2$ with a metric such that $\varphi$ is an isometry where defined.

Now $\varphi$ sends the stable (resp. unstable) manifold of $\Sigma_1$ to the stable (resp. unstable) manifold of $\Sigma_2$.

Upon replacing $U_0$ by a subset, we may assume that $U_0$ is contained in $S_1^{-1}([-\varepsilon,\varepsilon])$ for $\varepsilon$ small enough, and that $U_0$ is stable by the gradient flow of $S_1$ (restricted to $S_1^{-1}([-\varepsilon,\varepsilon])$).

We first extend $\varphi$ to $U_0\cup S_1^{-1}(-\varepsilon)$. Since $U_0\cap S_1^{-1}(-\varepsilon)$ is a neighbourhood of the "negative sphere bundle in $S_1^{-1}(-\varepsilon)$" that is the intersection of the unstable manifold of $\Sigma_0$ with $S_1^{-1}(\varepsilon)$, and since $S_1^{-1}(-\varepsilon)$ is diffeomorphic to the product of the "negative sphere bundle" with $E_1^+$, existence of an extension of $\varphi$ follows from the contractibility of the fibre of the map

(*) $$\mathrm{Diff}_+(S^k\times\mathbb{R}^q)\to\mathrm{Emb}_0(S^k\times D^q, S^k\times\mathbb{R}^q).$$

Here $\mathrm{Diff}_+(S^k \times \mathbb{R}^q)$ is the space of orientation preserving diffeomorphisms, and $\mathrm{Emb}_0$ is the set of embeddings inducing a degree one map on the spheres. That the above map is indeed a Serre fibration follows easily from the extension of isotopies principle, while the contractibility of the fibre is proved using Alexander's trick as follows: the fibre over the "standard" embedding is the space

$$\{\psi \in \mathrm{Diff}_+(S^k \times \mathbb{R}^q) \| \psi_{|S^k \times D^q} = i\}$$

where $i$ is the standard embedding of $S^k \times D^q$ into $S^k \times \mathbb{R}^q$. Set

$$\psi(x,\alpha) = (X(x,\alpha),\ A(x,\alpha)) \quad \text{and} \quad \psi_t(x,\alpha) = \left(X(x,t\alpha),\ \frac{1}{t} A(x,t\alpha)\right).$$

Then as $t$ goes to 0, $\psi_t(x,\alpha)$ goes to the identity map and we thus get our retraction of the fibre to a point.

Now, this means that the map (*) is a weak homotopy equivalence, which implies, using the connexity of $\mathrm{Emb}_0(S^k \times D^q, S^k \times \mathbb{R}^q)$ that any family of embeddings of $S^k \times D^q$ into $S^k \times \mathbb{R}^q$ (inducing a map of degree one on the spheres) extends to a family of diffeomorphisms of $S^k \times \mathbb{R}^q$. This is exactly what we need to be able to extend $\varphi$, since one can show that the embedding of the neighbourhood of the negative sphere (which is diffeomorphic to $S^k \times D^q$) into $S^{-1}(-\varepsilon) \simeq S^k \times \mathbb{R}^q$ induces a map of degree one on $S^k$.

We thus found our extension of $\varphi$ to a map

$$\tilde{\varphi} : U_0 \cup S_1^{-1}(-\varepsilon) \to U_1 \cup S_2^{-1}(-\varepsilon).$$

The next step is to extend $\tilde{\varphi}$ to $\bar{\varphi}$ defined on all $S_1^{-1}([-\varepsilon, \varepsilon])$. Let $\eta_t^1$ be the flow of the normalized fiber gradient of $S_1$, that is $S_1(\eta_t^2(b,\xi)) = S_1(b,\xi) + t$, which is well defined in $S_1^{-1}([-\varepsilon,\varepsilon]) - U_0$. Let $\eta_t^2$ be the analogous flow for $S_2$. We set for $(b,\xi) \in S_1^{-1}(-\varepsilon)$

$$\bar{\varphi}(\eta_t^1(b,\xi)) = \eta_t^2(\varphi(b,\xi)).$$

This definition coincides with $\varphi$ on $\partial U_0$, since by assumption $\varphi$ maps the fiber gradient flow of $S_1$ to the fiber gradient flow of $S_2$.

Now that $\bar{\varphi}$ is defined on $S_1^{-1}([-\varepsilon,\varepsilon])$ we may extend it to $\Phi : E_1 \to E_2$ by again following the flow of $\eta_t^1, \eta_t^2$. Our proof is now complete.

We may now conclude the proof of 1.5. This follows mainly from the construction of Laudenbach and Sikorav (cf. [L-S, Si1, Si2]).

Let $L$ be a Lagrange submanifold of $T^*B$ having a g.f.q.i. Then, according to [Si1], if $\psi_t$ is a Hamiltonian isotopy of $T^*B$, then $\varphi_1(L)$ has a g.f.q.i. given by

$$S_1(b,\xi,\eta) = \mathscr{A}(b,\xi) + S_0(\beta(b,\xi),\eta).$$

Here $\mathscr{A} = \mathscr{A}_{\varphi_t}$ is a function depending on $\varphi_t$ only, and $\beta : E \to B$ is <u>not</u> the canonical projection. We shall write $S_1 = \mathscr{A} * S_0$. Now it is clear that if $S_0$ is equivalent to $S_0'$, then $S_1' = \mathscr{A} * S_0'$ will be equivalent to $S_1$.

Consider now the case where $L = \varphi_1(O_B)$ and set $\mathscr{A}_t = \mathscr{A}_{\varphi_t^{-1}}$, $\mathscr{A}_{-t} = \mathscr{A}_{\varphi_t}$, then if $S_0, S_0'$ are g.f.q.i. for $L$, then

$$\mathscr{A}_1 * S_0 \quad \text{and} \quad \mathscr{A}_1 * S_0' \quad \text{are g.f.q.i.}$$

for the zero section, and are thus equivalent after stabilization. Therefore $\mathscr{A}_{-1} * \mathscr{A}_1 * S_0$ and $\mathscr{A}_{-1} * \mathscr{A}_1 * S_0'$ are equivalent after stabilization.

Now consider the family $\mathscr{A}_{-t} * \mathscr{A}_t * S_0$. It is a g.f.q.i. of

$$\varphi_t^{-1} \circ \varphi_t \circ \varphi_1(O_B) = \varphi_1(O_B),$$

that we denote by $R_t$. Similarly $R'_t = \mathscr{A}_{-t} * \mathscr{A}_t * S'_0$ is also a g.f.q.i. for $\varphi_1(O_B)$, and we just proved that $R_1$ and $R'_1$ are equivalent after stabilization.

But $R_1$ is equivalent to $R_0$ (and $R'_1$ to $R'_0$) because $R_t$ is a continuous family of g.f.q.i. for $\varphi_1(O_B)$. It is then easy to find a vector field generating a family of fiber preserving diffeomorphisms $\psi_t$ such that $R_t(\psi_t(b,\xi)) = R_0(b,\xi)$.

Thus $R_0$ is equivalent to $R'_0$ after stabilization, but $R_0 = \mathscr{A}_0 * \mathscr{A}_0 * S_0$ is easily seen to be a stabilization of $S_0$, and similarly $R'_0$ is a stabilization of $S'_0$. This concludes our proof.

## 2 Invariants for Lagrange submanifolds

In the next two sections, we suppose that generating functions are normalized in such a way that the indeterminacy of $S$ by a constant is removed. In most applications, we shall consider Lagrange submanifolds of $T^*B$ coinciding with the zero section in a certain subset of $B$. It is then natural to normalize our generating functions so that the critical value corresponding to this region is zero.

In this section $L$ will be a compact connected manifold, $H^*, H_*$ will be homology and cohomology with coefficients in some field $\mathbb{F}$, so that the Künneth formula can be written as

$$H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y),$$

$$H_*(X \times Y) \simeq H_*(X) \otimes H_*(Y).$$

Given $B$, we consider the following sets

$$\mathscr{L} = \{\text{Compact Lagrange submanifolds of } T^*B \text{ isotopic to the zero section}\}$$

and for $L$ in $\mathscr{L}$

$$G(L) = \{\text{g.f.q.i. of } L \text{ such that } E_\infty^- \text{ the negative bundle of } q_\infty \text{ is trivial}\}.$$

Note that if $S$ coïncides with $q_\infty$ at infinity, then for $\lambda$ large enough, the homotopy type of the pairs $(E^\lambda, E^\mu)$, $(E^\mu, E^{-\lambda})$ does not depend on $\lambda$. We may thus write $E^\infty, E^{-\infty}$ to denote $E^\lambda, E^{-\lambda}$ for $\lambda$ large enough. We remind the reader that, if we denote by $D(E_\infty^-)$, $S(E_\infty^-)$ the disk and sphere bundles associated to $E_\infty^-$, we have

$$H^*(E^\infty, E^{-\infty}) \simeq H^*(D(E_\infty^-), S(E_\infty^-)) \xrightarrow[\simeq]{T^{-1}} H^*(B),$$

where $T$ is the Thom isomorphism, shifting the grading by $d^- \overset{\text{def}}{=} \dim E_\infty^-$.

We are now ready to set

**Definition 2.1.** Let $(u, L) \in H^*(B) \times \mathscr{L}$ $(u \neq 0)$, we define a real number $c(u, L)$ as follows. Let $S \in G(L)$, and consider $Tu \in H^*(E^\infty, E^{-\infty})$. Then

$$c(u, L) = \inf\{\lambda \mid \text{the image of } Tu \text{ by the natural map } H^*(E^\infty, E^{-\infty}) \to H^*(E^\lambda, E^{-\infty}) \text{ is non zero}\}.$$

In view of the results of Sect. 1, $c(u, L)$ is indeed independent of $S$.

It is a classical result of Lusternik Schnirelman's theory that $c(u, L)$ is a critical value of $S$. Moreover we have

**Proposition 2.2.** *If* $u = v \cup w$ *then*

$$c(u, L) \geqq c(v, L)$$

*with equality only if* $w$ *induces a non zero class in a neighbourhood of*

$$K_c = \{\textit{critical points on the level } c(u, L)\}.$$

Note also that there is a one to one correspondence between critical points of $S$ and points of $L \cap B$ (where $B$ is identified with the zero section of $T^*B$). We shall elaborate on this later, meanwhile we get

**Corollary 2.3.** *Let* $\mu \in H^n(B)$ *be the orientation class of* $B$. *Then* $c(\mu, L) = c(1, L)$ *if and only if* $L = B$.

*Proof.* According to the proposition, $c(\mu, L) = c(1, L)$ implies that $\pi^*\mu$ induces a non zero cohomology class on a neighbourhood of $K_c$, or that $\mu$ induces a non zero cohomology class in a neighbourhood of $\pi(K_c) \subset L \cap B$. Because $\mu$ induces zero on $B - \{pt\}$, we must have $L \cap B = B$ that is $B \subset L$ hence $B = L$.

We now explain more precisely the relationship between the $c(\mu, L)$ and the points of $L \cap B$. Recall from [V1] that we may associate to a pair of Lagrange submanifolds $L_1, L_2$ of $T^*B$ and a pair $x, y$ of transverse intersection points of $L_1 \cap L_2$ an integer $m(x, y; L_1, L_2)$ and a real number $l(x, y; L_1, L_2)$ [denoted by $m(x, y)$ and $l(x, y)$ if there is no ambiguity]. The number $l(x, y)$ has the following simple definition: let $\gamma_1$ (resp. $\gamma_2$) be a path in $L_1$ (resp. $L_2$) connecting $x$ to $y$. Let $\gamma$ be the loop $\gamma_1 \gamma_2^{-1}$, then $l(x, y) = \int_\gamma p\,dq$. It is easy to see that our definition does not depend on the choices of $\gamma_1, \gamma_2$.

The definition of $m(x, y)$ can be found in [V1], we however point out that according to [V1] $m(x, y)$ is an integer modulo $\mu(L_1) \cdot H_1(L_1) + \mu(L_2) \cdot H_1(L_2)$ where $\mu(L_1), \mu(L_2)$ are the Maslov classes of $L_1, L_2$. Since we are only interested in these Lagrange submanifolds given by a generating function, their Maslov class vanishes, so $m(x, y)$ is indeed an integer. Now Proposition 5.2 of [V1, p. 370] and the above Proposition 2.2 yield

**Proposition 2.4.** *To each* $u \in H^*(B)$ *we can associate a point* $x_u$ *of* $L \cap B$ *such that*

(1) $$l(x_u, x_v) = c(u, L) - c(v, L)$$

(2) *if* $L$ *is transverse to* $B$, *then* $m(x_u, x_v) = \deg u - \deg v$.

This result is very useful, especially when combined with

**Proposition 2.5.** *If* $L_1, L_2$ *have generating functions* $S_1, S_2$ *defined on the same vector bundle, and such that* $\|S_1 - S_2\|_{C^0} \leqq \varepsilon$, *then for all* $u$ *in* $H^*(B)$ *we have* $|c(u, L_1) - c(u, L_2)| \leqq \varepsilon$.

*Proof.* The result is obvious, set $E_1^\lambda$ (resp. $E_2^\lambda$) to be

$$\{x \in E \mid S_1(x) \leqq \lambda\}$$

(resp. $\{x \in E \mid S_2(x) \leqq \lambda\}$) and remark that $E_2^{\lambda-\varepsilon} \subset E_1^\lambda \subset E_2^{\lambda+\varepsilon}$ and for $|\lambda|$ large enough all the inclusions are homotopy equivalences. Using the definition of $c(u, L)$ yields the result.

We are now ready to prove

**Proposition 2.6.** *Let $\psi_t$ be flow of a Hamiltonian preserving $T^*B-B$. Then for all $u$ in $H^*(L)$ we have*

$$c(u, \psi_t L) = c(u, L).$$

*Proof.* Note that for all $t$, $\psi_t L \cap B = L \cap B$, and for any two points $x, y$ in $L \cap B$, $l_t(x, y)$ [that we define to be $l(x, y;\ \psi_t L, 0_B)$] is really independent of $t$. If we assume that the g.f.q.i. of $L$ has finitely many critical values, then the set of their differences is finite. Thus $l(x, y)$ has values in a finite set.

On the other hand $c(u, \psi_t L)$ changes continuously with $t$, and according to 2.4 has its values in the set of $l(x, y)$. This implies that if the g.f.q.i. has only finitely many critical values, $c(u, \psi_t L)$ is a constant. The general case is obtained by perturbing a g.f.q.i. of $L$ so that the above argument applies, and use Proposition 2.5 to conclude.

Before we define the "composition" of Lagrange manifolds, and study the capacities of this "composition", we would like to state and prove some more elementary results. Let $\alpha$ be a homology class on $B$, then we can define $c(\alpha, L)$ by

$$c(\alpha, L) = \inf\{\lambda \mid \tilde{T}\alpha \in H_*(E^\infty, E^{-\infty}) \text{ is in the image of the map } H_*(E^\lambda, E^\infty) \to H_*(E^\infty, E^{-\infty})\}.$$

Here $\alpha \to \tilde{T}\alpha$ is the Thom isomorphism in homology.

Finally if $L$ is an immersed Lagrange submanifold of $T^*B$, $\bar{L}$ denotes the image of $L$ by $(q, p) \to (q, -p)$. Obviously, $\bar{L}$ is again a Lagrange submanifold. We now have

**Proposition 2.7.** *If $u \in H^q(B)$ and $\alpha \in H_{n-q}(B)$ are Poincaré dual to each other, then* $c(u, \bar{L}) = -c(\alpha, L)$.

*Proof.* If $S$ is a g.f.q.i. for $L$, then $-S$ is a g.f.q.i. for $\bar{L}$. Then

$$\bar{E}^\lambda = \{x \in E \mid -S(x) \leqq \lambda\} = E - E^{-\lambda}.$$

Thus $H^*(\bar{E}^\lambda, \bar{E}^{-\infty}) = H^*(E - E^{-\lambda}, E - E^\infty)$. This last group is isomorphic, by Alexander duality, to $H_*(E^\infty, E^{-\lambda})$.

$$\begin{array}{ccc}
H_*(E^{-\lambda}, E^{-\infty}) & \xrightarrow{\ \sim\ } & H^*(\bar{E}^\infty, \bar{E}^\lambda) \\
\downarrow & & \downarrow \\
\tilde{T}\alpha \in H_*(E^\infty, E^{-\infty}) & \xrightarrow{\ \sim\ } & H^*(\bar{E}^\infty, \bar{E}^{-\infty}) \ni Tu \\
\downarrow & & \downarrow \\
H_*(E^\infty, E^{-\lambda}) & \xrightarrow{\ \sim\ } & H^*(\bar{E}^\lambda, \bar{E}^{-\infty})
\end{array}$$

**Fig. 1**

Consider the diagram of Fig. 1, where the horizontal isomorphisms are given by Alexander duality, and the vertical ones from the homology (resp. cohomology) exact sequence of the triple $(E^\infty, E^{-\lambda}, E^{-\infty})$ [resp. $(\bar{E}^\infty, \bar{E}^\lambda, \bar{E}^{-\infty})$].

Note that the Alexander duality sends $\tilde{T}\alpha$ to $Tu$ if $u$ is the Poincaré dual of $\alpha$. We can then conclude our argument as follows:

If $\lambda < c(u, \bar{L})$ then the class $Tu$ goes to zero in $H^*(\bar{E}^\lambda, \bar{E}^{-\infty})$. This is equivalent to the fact that $\tilde{T}\alpha$ goes to zero in $H_*(E^\infty, E^{-\lambda})$, or else by exactness of the homology sequence of a triple, that $\tilde{T}\alpha$ is in the image of $H_*(E^{-\lambda}, E^{-\infty})$, which implies $-\lambda \geqq c(u, L)$ or $\lambda \leqq -c(\alpha, L)$. We just proved that $c(u, \bar{L}) \leqq -c(\alpha, L)$. The same argument yields the reversed inequality, hence our result.

**Corollary 2.8.** *If $\mu$ is the orientation class of $B$, then*

$$c(\mu,\bar{L}) = -c(1,L).$$

*Proof.* Let $\alpha \in H_0(L)$ be the Poincaré dual of $\mu$. Let $N = \dim E^-$, so that

$$\tilde{T}\alpha \in H_N(E^\infty, E^{-\infty}).$$

Because $\dim H_N(E^\infty, E^{-\infty}) = 1$ [due to $\dim H_0(L) = 1$], $c(\alpha, L)$ can also be defined as

$$\inf\{\lambda \mid H_N(E^\lambda, E^{-\infty}) \to H_N(E^\infty, E^{-\infty}) \text{ is non zero}\}.$$

Since we are dealing with coefficients in a field, $\mathbb{F}$,

$$H_N(E^\lambda, E^{-\infty}) \simeq \mathrm{Hom}(H^N(E^\lambda, E^{-\infty}), \mathbb{F})$$

and the map $H_N(E^\lambda, E^{-\infty}) \to H_N(E^\infty, E^{-\infty})$ is the transpose of the map

$$H^N(E^\infty, E^{-\infty}) \to H^N(E^\lambda, E^{-\infty})$$

which is non zero if and only $\lambda \geqq c(1,L)$. Thus $c(\alpha, L) = c(1,L)$, and the conclusion follows from 2.7.

## 3 Composition of Lagrange submanifolds

In the coming sections we shall deal with Lagrange submanifolds of $T^*B$ not necessarily isotopic to the zero section, but still possessing a g.f.q.i., $S$.

We shall first define for a pair of Lagrange submanifolds $L_1, L_2$ of $T^*B$ the Lagrange manifold $L_1 \# L_2$.

**Definition 3.1.** Let $L_1, L_2$ be two Lagrange submanifold of $T^*B$, and assume $L_1 \times L_2$ to be transverse to $T^*_\Delta(B \times B)$ [= the restriction to the diagonal $\Delta$ of $T^*(B \times B)$].

Then, since $T^*_\Delta(B \times B)$ is a coisotropic manifold the reduction of which can be identified with $T^*B$, $L_1 \times L_2$ has a reduction to $T^*B$ that we denote by $L_1 \# L_2$.

Note that $L_1 \# L_2$ could be written as $L_1 + L_2$, since

$$L_1 \# L_2 = \{(q,p) \in T^*L \mid p = p_1 + p_2,\ (q,p_1) \in L_1,\ (q,p_2) \in L_2\}$$

and $L_1 \# \bar{L}_2$ could be written $L_1 - L_2$. Note that $L_1 - L_1$ is not the zero section $O_B$ unless the projection $L_1 \to B$ is one to one.

The proof of the following result is left to the reader.

**Proposition 3.2** *Let $S_1, S_2$ be g.f.q.i. for $L_1$ and $L_2$. Set*

$$S_1 \# S_2(x,\xi,\eta) = S_1(x,\xi) + S_2(x,\eta).$$

*Then $S_1 \# S_2$ is a g.f.q.i. for $L_1 \# L_2$.*

The main result of this section can now be stated as

**Proposition 3.3.**

$$c(u \cdot v, S_1 \# S_2) \geqq c(u, S_1) + c(v, S_2)$$

*for any $u, v \in H^*(L)$.*

*Proof.* Let $S_1$ be defined on $E_1$, $S_2$ on $E_2$ and $S_3=S_1 \# S_2$ on $E_3=E_1 \oplus E_2 = E_1 \times_B E_2$. Then

$$E_3^\nu = \bigcup_{\lambda+\mu=\nu} E_1^\lambda \times_B E_2^\mu . \quad \square$$

$$\begin{array}{ccc}
H^*(E_3^\infty, E_3^{-\infty}) \simeq & H^*(E_1^\infty, E_1^{-\infty}) \otimes_{H^*(B)} H^*(E_2^\infty, E_2^{-\infty}) & \\
\downarrow & \downarrow & \\
H^*(E_3^\nu, E_3^{-\infty}) & H^*(E_1^\lambda, E_1^{-\infty}) \otimes_{H^*(B)} H^*(E_2^\mu, E_2^{-\infty}) & \\
\searrow & \swarrow & \\
 & H^*(E_1^\lambda \times_B E_2^\mu, E_1^{-\infty} \times_B E_2^{-\infty}) &
\end{array}$$

**Fig. 2**

$$\begin{array}{ccc}
X \times_B Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & B
\end{array}$$

**Fig. 3**

$$\begin{array}{ccccc}
 & & X \times_B Y & & \\
 & \nearrow & & \searrow & \\
Z^\nu = \bigcup_{\lambda+\mu=\nu} X^\lambda \times_B Y^\mu & & & & X \times Y \\
 & \searrow & & \nearrow & \\
 & & \bigcup_{\lambda+\mu=\nu} X^\lambda \times Y^\mu & &
\end{array}$$

**Fig. 4**

Consider the commutative diagram of Fig. 2, where all arrows are induced by natural maps, except for the lower right one

$$H^*(E_1^\lambda, E_1^{-\infty}) \otimes_{H^*(B)} H^*(E_2^\mu, E_2^{-\infty}) \to H^*(E_1^\lambda \times_B E_2 \mu, E_1^{-\infty} \times_B E_2^{-\infty}) .$$

This last map is defined as follows. Let $p_1, p_2$ be the projections of $E_3$ on $E_1, E_2$. Then we have a map $p_1^* \otimes p_2^*$

$$H^*(E_1^\lambda, E_1^{-\infty}) \otimes H^*(E_2^\mu, E_2^{-\infty}) \to H^*(E_1^\lambda \times_B E_2^\mu, E_1^{-\infty} \times_B E_2^{-\infty})$$

and it is clear that for $\beta \in H^*(B)$, $x_1 \in H^*(E_1^\lambda, E_1^{-\infty})$, $x_2 \in H^*(E_2^\mu, E_2^{-\infty})$, $\beta x_1 \otimes x_2 - x_1 \otimes \beta x_2$ is in the kernel of $p_1^* \otimes p_2^*$. Thus $p_1^* \otimes p_2^*$ can in fact be defined on $H^*(E_1^\lambda, E_1^{-\infty}) \otimes_{H^*(B)} H^*(E_2^\mu, E_2^{-\infty})$.

We now need the

**Lemma 3.4.** *Let $X, Y$ be total spaces of fiber bundles over $B$. Then*

$$H^*(X \times_B Y) \simeq H^*(X) \otimes_{H^*(B)} H^*(Y) .$$

*Consider now two increasing filtrations $(X^\lambda), (Y^\mu)$ of $X$ and $Y$, and let*

$$Z^\nu = \bigcup_{\lambda+\mu=\nu} X^\lambda \times_B Y^\mu .$$

*If a cohomology class $x \otimes y$ in $H^*(X \times_B Y)$ induces a non zero class on $H^*(Z^\nu)$, then there exists $\lambda, \mu$ with $\lambda+\mu=\nu$ such that $x \otimes y$ is non zero in $H^*(X^\lambda) \otimes H^*(Y^\mu)$.*

The Thom isomorphism just tells us that $H^*(E^\infty, E^{-\infty})$ is a free $H^*(B)$ module. By applying the lemma to our situation, we see that there exist $\lambda, \mu$ with $\lambda+\mu=\nu$

such that if the image of $T(uv) \in H^*(E_3^\infty, E_3^{-\infty})$ in $H^*(E_3^\nu, E_3^{-\infty})$ is non zero, then its image in

$$H^*(E_1^\lambda \times_B E_2^\mu, E_1^{-\infty} \times_B E_2^{-\infty})$$

is also non zero. Using the diagram of Fig. 2, this implies that the image of $Tu \otimes Tv$ in

$$H^*(E_1^\lambda, E_1^{-\infty}) \otimes_{H^*(B)} H^*(E_2^\mu, E_2^{-\infty})$$

is non zero, thus $\lambda \geqq c(u, L_1)$, $\mu = c(v, L_2)$. This shows that

$$c(uv, L_1 \# L_2) \geqq c(u, L_1) + c(v, L_2).$$

We now complete our proof by proving Lemma 3.4. The first statement is proved by either using the Eilenberg-Moore spectral sequence associated to the diagram of Fig. 3, or in our case, the fact that if $T_E$ denotes the Thom class associated to $(E^\infty, E^{-\infty})$, then we have $T_{E_1 \oplus E_2} = T_{E_1} \otimes T_{E_2}$. For the second part, we use the commutative diagram of Fig. 4 and the first part of the lemma. As a consequence of this, the map $X \times_B Y \to X \times Y$ induces the map

$$H^*(X) \otimes H^*(Y) \to H^*(X) \otimes_{H^*(B)} H^*(Y)$$

which is surjective. Thus if a given class in $H^*(X \times_B Y)$ induces a non zero class in $H^*(Z)$, the same will be true for some class $x \otimes y$ in $H^*(X \times Y)$. Using the commutativity of the diagram of Fig. 4, this implies that $x \otimes y$ has non zero image in $H^*\left(\bigcup_{\lambda+\mu=\nu} X^\lambda \times Y^\mu\right)$, we now show that this implies that $x \otimes y$ has non zero image in $H^*(X^\lambda \times Y^\mu)$ for some $\lambda, \mu$ with $\lambda + \mu = \nu$.

This is in fact readily checked, let

$$\lambda(x) = \inf\{\lambda \mid x \text{ is non zero in } H^*(X^\lambda)\},$$

$$\mu(x) = \inf\{\mu \mid x \text{ is non zero in } H^*(X^\mu)\}.$$

We claim that $\lambda(x) + \mu(y) \geqq \nu$. Assume this is not so, we can then find cocycles $a$ and $b$ with support in $X - X^{\lambda(x)}$, $Y - Y^{\mu(y)}$, and cocycles $\alpha, \beta$ such that

$$x = [a + d\alpha], \qquad y = [b + d\beta]$$

hence

$$x \otimes y = a \otimes b + d(-).$$

Now the cocycle $a \otimes b$ vanishes on $X^\lambda \times Y^\mu$ if either $\lambda < \lambda(x)$ or $\mu < \mu(y)$, in particular if $\lambda + \mu = \nu$ we are in one of the two cases, and $a \otimes b$ vanishes. Since $Z = \bigcup_{\lambda+\mu=\nu} X^\lambda \times Y^\mu$, $a \otimes b$ vanishes on $Z$, a contradiction.

**Proposition 3.5.** $c(u, \psi(L)) = c(u, L - \psi^{-1}(O_B))$.

*Proof.* We now consider $c(u, \psi_t(L) - \psi_t\psi_1^{-1}(0_B))$. This takes values in the set of $\int_{\gamma_{1,t}} p\,dq - \int_{\gamma_{2,t}} p\,dq$, where $\gamma_{1,t}$ and $\gamma_{2,t}$ are paths connecting two points $x_t, y_t$ in

$$\psi_t(L) \cap \psi_t\psi_1^{-1}(0_B) = \psi_t(L \cap \psi_1^{-1}(0_B)),$$

and $\gamma_{1,t}$ (resp. $\gamma_{2,t}$) is in $\psi_t(L) - \psi_t\psi_1^{-1}(0_B)$ (resp. $0_B$).

Now we would like to take for $\gamma_{1,t}$, $\psi_t\psi_1^{-1}(\gamma_{1,1}) - \psi_t\psi_1^{-1}(\gamma_{2,1})$ but this expression is meaningless since the two paths we are trying to substract do not necessarily have the same projection. However it is easy to get around this difficulty as follows.

Let $S_{1,t}(b,\xi)$ [resp. $S_{2,t}(b,\eta)$] be a g.f.q.i. for $\psi_t\psi_1^{-1}(\psi_1(L))$ [resp. $\psi_t\psi_1^{-1}(0_B)$], and $x_t=(q_t,p_t)$, $y_t=(q_t',p_t')$. Then $(q_t,0)$ and $(q_t',0)$ are in

$$(\psi_t\psi_1^{-1}(\psi_1(L))-\psi_t\psi_1^{-1}(0_B))\cap 0_B .$$

Now $l((q_t,0),(q_t',0))$ is given by

$$[S_{1,t}(q_t,\xi_t)-S_{2,t}(q_t,\eta_t)]-[S_{1,t}(q_t',\xi_t')-S_{2,t}(q_t',\eta_t')]$$

where $(q_t,\xi_t,\eta_t)$ and $(q_t',\xi_t',\eta_t')$ correspond to $(q_t,0)$ and $(q_t',0)$. But the above expression can be reordered as

$$[S_{1,t}(q_t,\xi_t)-S_{1,t}(q_t',\xi_t')]-[S_{2,t}(q_t,\eta_t)-S_{2,t}(q_t',\eta_t')] .$$

This can be read as the difference of $\int_{\psi_t\psi_1^{-1}(\gamma_{1,1})} pdq$, and $\int_{\psi_t\psi_1^{-1}(\gamma_{2,1})} pdq$, and it is now clear that $l((q_t,0),(q_t',0))=\int_{\gamma_{2,1}\circ\gamma_{1,1}^{-1}} pdq$ is indeed independent of $t$. □

**Corollary 3.6.** $c(uv,\psi(L))\geqq c(u,L)+c(v,\psi(0_B))$.

*Proof.* According to the proposition we just proved, we have:

$$c(uv,\psi(L))=c(uv,L-\psi(0_B)) .$$

We now apply Proposition 3.3 to get

$$c(uv,\psi(L))\geqq c(u,L)+c(v,\overline{\psi^{-1}(0_B)}) .$$

Applying 3.5 once more shows that

$$c(v,\psi(0_B))=c(v,0_B-\psi^{-1}(0_B))=c(v,\overline{\psi^{-1}(0_B)}) ,$$

thus concluding our proof. □

## 4 Applications to Hamiltonian diffeomorphisms

In this section we consider the following subgroups of $\mathrm{Diff}_\omega(\mathbb{R}^{2n})$, the group of symplectic diffeomorphisms of $\mathbb{R}^{2n}$:

$\mathscr{H}^0(\mathbb{R}^{2n})$: the group of time one maps of the Hamiltonian flow associated to a time dependent, compact supported Hamiltonian $H(x,t)$

$\mathscr{H}(U)$: the subgroup of $\mathscr{H}^0(\mathbb{R}^{2n})$ obtained by imposing to the support of $H(x,t)$ to be in $U\times[0,1]$.

*Remark.* If $H(t,x)$ is a Hamiltonian, generating a flow $\psi_t$ we will denote by $\widetilde{\mathrm{supp}}(\psi_t)$ the support of $H$. We point out that it is generally bigger than the support of the isotopy (i.e. $\{x\mid\exists t,\psi_t(x)\neq x\}$). For instance take $H$ to be one in a neighbourhood of the origin, and 0 outside a compact set. Then the neighbourhood of the origin is in $\widetilde{\mathrm{supp}}(\psi_t)-\mathrm{supp}(\psi_t)$.

The group $\mathscr{H}(B^{2n}(r))$ has obvious $C^k$ $(k\geqq 0)$ topologies, and $\mathscr{H}^0(\mathbb{R}^{2n})$ will have the obvious topology that makes it a limit of $\mathscr{H}(B^{2n}(r))$ as $r$ goes to infinity. Let us remind the reader that as a result, a continuous map from a compact set into $\mathscr{H}^0(\mathbb{R}^{2n})$ can be factored through $\mathscr{H}(B^{2n}(r))$ for $r$ large enough.

We now consider, for $\psi \in \overline{(\mathbb{R}^{2n})}$, its graph $\Gamma_\psi$, a Lagrange submanifold of $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$. For $r$ large enough, $\Gamma_\psi$ coincides with the diagonal $\Delta$ outside $B^{2n}(r) \times B^{2n}(r)$. We can identify $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ with $T^*\Delta$ through the map

$$(q,p,Q,P) \to \left(\frac{q+Q}{2}, \frac{p+P}{2}, P-p, q-Q\right).$$

Now the image of $\Gamma_\psi$ in $T^*\Delta$, $\Gamma_\psi$ coïncides with the zero section outside a compact set. If we consider $\Delta \simeq \mathbb{R}^{2n}$ as $S^{2n} - \{P\}$, where $P$ is the "north pole", we get an embedding $T^*\Delta \rightarrow T^*S^{2n}$, and the image of $\Gamma_\psi$ coincides with the zero section in a neighbourhood of $P$, so that we can make this image closed by adding the point $(P,0)$. We thus get a Lagrange submanifold $\tilde{\Gamma}_\psi$ of $T^*S^{2n}$, such that there is a neighbourhood $U$ of $P$ in $S^{2n}$ with $\tilde{\Gamma}_\psi \cap T^*U = U$. We shall normalize generating functions by assuming that $S(P,0)=0$.

We may now set

**Definition 4.1.** Let $H^*(S^{2n}) = \mathbb{F} \cdot 1 \oplus \mathbb{F} \cdot \mu$, we set

$$c_-(\psi) = -c(\mu, \tilde{\Gamma}_\psi),$$
$$c_+(\psi) = -c(1, \tilde{\Gamma}_\psi),$$
$$\gamma(\psi) = c_+(\psi) - c_-(\psi).$$

The results of Sect. 2 have obvious translations to $c_\pm(\psi), \gamma(\psi)$, however we summarize the main facts together with some improvements, and new results.

**Proposition 4.2.**

(1) $$c_-(\psi) \leqq 0 \leqq c_+(\psi), \quad \textit{with} \quad c_+(\psi) = c_-(\psi) = 0$$

*if and only if* $\psi = \mathrm{Id}$

(2) $$c_+(\psi^{-1}) = -c_-(\psi).$$

(3) *Let $U$ be a compact set in $\mathbb{R}^{2n}$, $\psi_t$ be an isotopy in $\mathscr{H}(U)$, generated by $H$, and $z$ be any point in $\mathbb{R}^{2n} - U$.*

*There are fixed points of $\psi, x_+, x_-$ in $U$ such that if $\gamma_+, \gamma_-$ are paths connecting $x_+$ to $z$ and $x_-$ to $z$, and $g_+ = \gamma_+ \circ \psi(\gamma_+^{-1})$, $g_- = \gamma_- \circ \psi(\gamma_-^{-1})$ then*

$$c_+(\psi) = l(x_+, z; \psi) = -\int_{g_+} pdq = -\int_{\psi_t(x_+)} pdq - Hdt,$$
$$c_-(\psi) = l(x_-, z; \psi) - \int_{g_-} pdq = -\int_{\psi_t(x_-)} pdq - Hdt.$$

*Moreover if the fixed points $x_+, x_-$ are nondegenerate, we have with the notations of* [V2],

$$i_{cz}(x_+) = 2n, \quad i_{cz}(x_-) = 0.$$

*Remark.* $l(x,y;\psi)$ is obviously defined as $l((x,x),(y,y);\Gamma_\psi,\Delta)$ (cf. Sect. 2).

*Proof.* (1) Let $S$ be a g.f.q.i. for $\tilde{\Gamma}_\psi$ obtained from one for $\Gamma_\psi$, then

$$S(\infty, \xi) = |\xi_+^*|^2 - |\xi_-^*|^2.$$

Let $E_{\{P\}}$ be the fiber of $E$ over the north pole identified to the point at infinity, then we have inclusions

$$(E^0_{\{P\}}, E^{-\infty}_{\{P\}}) \hookrightarrow (E^0; E^{-\infty})$$

hence a commutative diagramm

$$\begin{array}{ccc} H^*(E^0_{\{P\}}, E^{-\infty}_{\{P\}}) & \longleftarrow & H^*(E^0, E^{-\infty}) \\ {\scriptstyle j^*T}\uparrow & & \uparrow{\scriptstyle j^*T} \\ H^*(\{P\}) & \longleftarrow & H^*(S^{2n}) \end{array}$$

where $T$ is the Thom isomorphism

$$H^*(S^{2n}) \to H^*(E^\infty, E^{-\infty})$$

(resp. $H^*(\{\infty\}) \to H^*(E^\infty_{\{P\}}, E^{-\infty}_{\{\infty\}})$) and $j^*$ is induced by the inclusion map $j:(E^0, E^{-\infty}) \to (E^\infty, E^{-\infty})$ (resp. ...).

Now it is easy to see that $j^*T$ is an isomorphism, and since the image of $1 \in H^*(S^{2n})$ is 1 in $H^*(\{\infty\})$ we see that $j^*T(1_{S^{2n}}) \neq 0$ hence $c_-(\psi) = c(1, \tilde{\Gamma}_\psi) \leqq 0$.

The inequality $c_+(\psi) \geqq 0$ now follows from (2) which is a translation of Corollary 2.8 since $\tilde{\Gamma}_\psi = \tilde{\Gamma}_{\psi^{-1}}$. Finally the left side equality in (3) is a restatement of Proposition 2.4. The equality

$$\int_g pdq = \int_{\psi_t(x)} pdq - Hdt$$

is proved as follows. Let $\Psi: K = [0,1]^2 \to \mathbb{R}^{2n}$ be the map given by $\Psi(t,\theta) = \psi_t(\gamma(\theta))$. Then $\int_{\Psi(K)} d(pdq - Hdt) = 0$ since for each $\theta, t \to \Psi(t,\theta)$ is an tangent to the kernel of $d(pdq - Hdt)$, and $\Psi(K)$ is foliated by such curves. Now apply Stoke's theorem, and decomposing $\partial K$ as $\{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$, we get

$$\begin{aligned} \int_g pdq &= \int_{\Psi(\{0,1\}\times[0,1])} pdq - Hdt = \int_{\Psi[(0,1]\times\{0,1\})} pdq - Hdt \\ &= \int_{\psi_t(x)} pdq - Hdt - \int_{\psi_t(z)} pdq - Hdt = \int_{\psi_t(x)} pdq - Hdt. \end{aligned}$$

The last equality holds because $\psi_t(z) \equiv z$ and $H(t,z) \equiv 0$. □

**Corollary 4.3.** *Let $\varphi$ be a conformal map isotopic to the identity in the set of conformal maps i.e. $\varphi = \varphi_1$, $\varphi_0 = \mathrm{id}$ with $\varphi_t^*\omega = \lambda(t)\omega$. Then, for any symplectic (not necessarily compact supported) diffeomorphism of $\mathbb{R}^{2n}$, we have*

$$c_+(\varphi\psi\varphi^{-1}) = \lambda^2(1)c_+(\psi), \quad c_-(\varphi\psi\varphi^{-1}) = \lambda(1)^2 c_-(\psi),$$

*and*

$$\gamma(\varphi\psi\varphi^{-1}) = \lambda^2(1)\gamma(\psi).$$

*Proof.* If $x, y$ are fixed points of $\psi$, then $\varphi_t(x), \varphi_t(y)$ are fixed points of $\varphi_t\psi\varphi_t^{-1}$, and

$$l(\varphi_t(x), \varphi_t(y)) = \lambda(t)^2 l(x,y)$$

so $\frac{1}{\lambda(t)^2} c_+(\varphi_t\psi\varphi_t^{-1})$ has values in a totally discontinuous set, hence is independent of $t$. The same holds for $c_-$ and $\gamma$. □

One of the main tools in the applications of $c_-, c_+, \gamma$ is given by

**Proposition 4.4.**

$$c_+(\varphi\psi) \leqq c_+(\varphi) + c_+(\psi),$$

$$c_-(\varphi\psi) \geqq c_-(\varphi) + c_-(\psi),$$

$$\gamma(\varphi\psi) \leqq \gamma(\varphi) + \gamma(\psi).$$

*Proof.* It follows from Corollary 3.6 which states

$$c(uv, \Psi(L)) \leqq c(u, L) + c(v, \overline{\Psi^{-1}(0_B)})$$

applied to the extension of $\Psi = \psi \times \mathrm{Id}$ to $T^*S^{2n}$ and $L = \tilde{\Gamma}_\varphi$. □

**Corollary 4.5.** *Let $\varphi_t$ be an isotopy in $\mathscr{H}^0(U)$, and $\psi$ an element in $\mathscr{H}^0(\mathbb{R}^{2n})$ such that $\psi(U) \cap U = \emptyset$. Then we have*

$$c_+(\psi\varphi_1) = c_+(\psi), \qquad c_-(\psi\varphi_1) = c_-(\psi).$$

*As a result we get $c_+(\varphi) \leqq \gamma(\psi)$ and $-c_-(\varphi) \leqq \gamma(\psi)$.*

*Proof.* Let $x$ be a fixed point of $\psi\varphi_t$. If $x$ is such a point, then $x$ is not in $U$ for otherwise $\varphi_t(x)$ would be in $U$ thus contradicting the assumption that $\psi(U) \cap U = \emptyset$. Since $\varphi_t \equiv \mathrm{id}$ outside $U$, we have that $x$ must be a fixed point of $\psi$. Now let $x, y$ be two fixed points of $\psi \circ \varphi_t$ (and thus of $\psi$). Then the number $l(x, y; \psi\phi_t)$ is independent of $t$. We choose a path $\gamma_1$ joining $x$ to $y$, then

$$\gamma_1 \circ \psi\varphi_t(\gamma_1^{-1}) = \gamma_1 \circ \psi(\gamma_1^{-1}) \circ \psi(\gamma_1) \circ \psi\phi_t(\gamma_1^{-1}) = \gamma_1 \circ \psi(\gamma_1^{-1}) \circ \psi(\gamma_1 \circ \phi_t(\gamma_1^{-1})).$$

First observe that the area of the closed loop $\psi(\gamma_1 \circ \phi_t(\gamma_1^{-1}))$ is the same as that of $\gamma_1 \circ \phi_t(\gamma_1^{-1})$ which is equal to $l(x, y; \phi_1)$, it is enough to show that this last quantity is zero. But by the proof of 4.2, $l(x, y; \phi_t)$ is also given by

$$\int_{\phi_t(x)} pdq - Hdt - \int_{\phi_t(y)} pdq - Hdt.$$

Since $\phi_t(x) = x$; $\phi_t(y) = y$ and $H(t, x) = H(t, y) = 0$, the area of $\gamma_1 \circ \phi_t(\gamma_1^{-1})$ vanishes. The area we are looking for is therefore the area of $\gamma_1 \circ \psi(\gamma_1^{-1})$, that is $l(x, y; \psi)$, and this proves our claim.

Our now classical argument shows that $c_+(\psi \circ \varphi_t)$ is independent of $t$, and the same holds for $c_-, \gamma$. Now we apply 4.4

$$c_+(\psi\varphi) + c_+(\psi^{-1}) \geqq c_+(\varphi)$$

since $c_+(\psi\varphi) = c_+(\psi)$, $c_+(\psi^{-1}) = -c_-(\psi)$ we get

$$c_+(\varphi) \leqq \gamma(\psi).$$

The other inequality follows similarly. □

**Proposition 4.6.** *Let $H_1(t, x)$, $H_2(t, x)$ be two compact supported Hamiltonians, $\psi_1$ and $\psi_2$ the time one maps of the associated flow. Assume $H_1(t, x) \leqq H_2(t, x)$ for all $x$ and $t$. Then $c_+(\psi_1) \leqq c_+(\psi_2)$ and $c_-(\psi_1) \leqq c_-(\psi_2)$.*

*Proof.* Let $K_1(x, y, t)$ be defined on $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ by $K_1(x, y, t) = H_1(y, t)$, and the same for $K_2$. Then the time one flow of the Hamiltonian vector field associated to $K_1$ is $(\mathrm{Id}, \psi_1)$ and so sends the diagonal to the graph of $\psi_1$. $K_1$ has an obvious extension to $T^*S^{2n}$ and his time one flow sends the zero action to $\tilde{\Gamma}_{\psi_1}$. The same holds for $K_2$, and of course $K_1 \leqq K_2$. We consider $H_u(t, x)$ to be a linear homotopy from $H_1$ to $H_2$ and $K_u$ the corresponding function on $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$. We first prove:

**Lemma 4.7.** *Let $S_\lambda$ be a continuous one parameter family of functions, and let $c(\lambda) = c(u, S_\lambda)$ be a critical value obtained by minmax. We assume that: $dS_\lambda(x) = 0$ implies $\frac{\partial}{\partial \lambda} S_\lambda(x) \geqq 0$.*

*Then* $c(\lambda)$ *is non decreasing.*

*Proof.* We first make the stronger assumption that for $x$ a critical point of $S_\lambda$, we have $\frac{\partial}{\partial\lambda}S_\lambda(x)>0$. We also assume that the family $S_\lambda$ is generic, i.e. each $S_\lambda$ has Morse critical points with distinct critical values, except for a finite number of values of $\lambda$, that we shall denote by $\lambda_1<\lambda_2<\ldots<\lambda_k$. Now if $\lambda\in]\lambda_j,\lambda_{j+1}[$ then we may follow the unique critical point with critical value $c(\lambda)$, let $x_\lambda$ be this critical point. Then

$$\frac{d}{d\lambda}c(\lambda)=\frac{d}{d\lambda}(S_\lambda(x_\lambda))=\frac{\partial}{\partial\lambda}S_\lambda(x_\lambda)+dS_\lambda(x_\lambda)\frac{d}{d\lambda}x_\lambda.$$

Because $x_\lambda$ is a critical point of $S_\lambda$ the second term is zero while the first one is positive by assumption. Thus $c$ is a continuous function with strictly positive derivative except at finitely many points. It is thus increasing. It is now easy to conclude in the general case. First the genericity assumption on the family is easily removed by approximation in the $C^1$ topology. This will not destroy the stronger hypothesis we made at the beginning of the proof, since the critical points of the perturbed family are in a neighbourhood of the critical point of the old one. If we now consider a sequence of generic families converging to the original one, since $c(u,S)$ depends continuously on $S$ in the $C^1$ topology, we get that $c(t)$ is non decreasing. Finally the stronger assumption we made is easily removed by replacing $S_\lambda$ by $S_\lambda+\varepsilon\lambda$ for $\varepsilon$ arbitrarily small. □

We may now prove the proposition. Let $\psi_t^\lambda(x_\lambda)$ be the flow of $H_\lambda(t,x)$, and $\phi_\lambda(t)=\psi_t^\lambda(x_\lambda)$. According to 4.2, we have for a critical point $x_\lambda$ of a g.f.q.i. $S_\lambda$ of $L_\lambda$, the graph of $\psi_1^\lambda$, that

$$S_\lambda(x_\lambda)=-\int_{S^1}\phi_\lambda^*[pdq-H_\lambda dt].$$

It is easy to check that

$$\frac{\partial}{\partial\lambda}S_\lambda(x_\lambda)=\int_{S^1}\frac{\partial}{\partial\lambda}H_\lambda(t,\phi_\lambda(t))dt\geqq 0. \quad \square$$

The above result has two applications. The first is to generalize the notion of time one flow of a nonnegative Hamiltonian terms of the invariants that we defined. To begin with, we have:

**Corollary 4.8.** *If* $\psi$ *is the time one map of the flow generated by a nonnegative Hamiltonian* $H(t,x)$, *then* $c_-(\psi)=0$. *If* $H(t,x)\geqq\varepsilon>0$ *for* $(t,x)$ *in* $]t_0,t_1[\times B(\eta)$, *then* $c_+(\psi)>0$.

*Proof.* Because $H\geqq 0$, $c_-(\psi)\geqq c_-(\mathrm{Id})=0$ and since $c_-(\psi)\leqq 0$ we get $c_-(\psi)=0$.

We now prove that if $H(t,x)\geqq\varepsilon>0$ for $(t,x)$ in $]t_0,t_1[\times B(\eta)$, then $c_+(\psi)>0$. This will conclude our proof.

For this it is enough to construct some nonnegative Hamiltonian $H_0$ such that $H_0$ has support in $]t_0,T_1[\times B(\eta)$, is bounded by $\varepsilon$, and such that $\varphi$, the time one flow associated to $H_0$, is not the identity. Indeed, we will have, according to the first part of our corollary, $c_-(\varphi)=0$. Since $\varphi\neq\mathrm{id}$, $c_+(\varphi)>0$, and because $H_0\leqq H$, $0<c_+(\varphi)<c_+(\psi)$. □

From 4.7 we see that the property of being the time one flow for a nonnegative Hamiltonian implies $c_-(\psi)=0$. We can thus define a partial order relation, that we denote by $\prec$:

**Definition 4.9.** For $\varphi, \psi$ in $\mathscr{H}^0(\mathbb{R}^{2n})$ we write $\varphi \prec \psi$ if and only if $c_-(\psi\varphi^{-1})=0$.

**Proposition 4.10.**

(1) $$\mathrm{Id} \prec \psi \Leftrightarrow \psi^{-1} \prec \mathrm{Id},$$

(2) $$\psi_1 \prec \psi_2 \Leftrightarrow \varphi\psi_1 \prec \varphi\psi_2,$$

(3) $$\varphi_1 \prec \varphi_2 \quad \textit{and} \quad \psi_1 \prec \psi_2 \Rightarrow \varphi_1\psi_1 \prec \varphi_2\psi_2.$$

*Proof.* Clearly (3)→(2)→(1). Now $\varphi_1\psi_1 \prec \varphi_2\psi_2$ means $c_-(\varphi_2\psi_2\psi_1^{-1}\varphi_1^{-1})=0$. But

$$c_-(\varphi_2\psi_2\psi_1^{-1}\varphi_1^{-1}) = c_-(\varphi_1^{-1}\varphi_2\psi_2\psi_1^{-1}) \geqq c_-(\varphi_1^{-1}\varphi_2) + c_-(\psi_2\psi_1^{-1})$$

since $\varphi_2 \succ \varphi_1$, $\psi_2 \succ \psi_1$ the right hand side is zero, and so is the left hand side. □

Now let $\psi \succ \mathrm{Id}$, then $\psi^k \prec \mathrm{Id}$ for all positive $k$, moreover $c_+(\psi^k) \leqq c_+(\psi^{k+1}) + c_+(\psi^{-1})$ since $c_+(\psi^{-1}) = -c_-(\psi) = 0$ we get that the sequence $c_+(\psi^k)$ is increasing. Thus either $c_+(\psi^k)$ goes to infinity, or it converges to some finite value.

In fact we see that $c_+(\psi^k)$ is bounded by the following argument. Denote by $U$ the support of $\psi_t$, and let $\varphi$ be an element of $\mathscr{H}^0(\mathbb{R}^{2n})$ such that $\varphi(U) \cap U = \emptyset$. By 4.5, we have $\gamma(\varphi) \geqq c_+(\psi^k)$ hence our result. In fact we just showed that

$$\sup\{c_+(\psi) \mid \mathrm{support}(\psi) \subset U\} \leqq \inf\{\gamma(\varphi) \mid \varphi(U) \cap U = \emptyset\}.$$

We now set

**Definition 4.11.** We define $c(U)$ as $\sup\{c_+(\psi) \mid \mathrm{support}(\psi) \subset U\}$ whenever $U$ is an open bounded set of $\mathbb{R}^{2n}$, and $\gamma(U)$ as $\inf\{\gamma(\psi) \mid \psi(U) \cap U = \emptyset\}$.

We may thus summarize our findings in

**Proposition 4.12.** (1) *If* $\mathrm{support}(\phi) \subset U$, *and* $\psi(U) \cap U = \emptyset$

$$c_+(\phi) \leqq c(U) \leqq \gamma(U) \leqq \gamma(\psi).$$

(2) *If* $\mathrm{Id} \prec \psi$ *the increasing sequence* $c_+(\psi^k)$ *has a limit, denoted by* $\bar{c}_+(\psi)$.

We now get a result on periodic points of compactly supported Hamiltonian systems.

Note that if $x_+(k)$, $x_-(k)$ are fixed points of $\psi^k$ given by Proposition 4.2, then $x_+(k) = x_+(kl)$ implies that $c_+(\psi^{kl}) = lc_+(\psi^k)$, since if $\gamma$ is a path from $x_+(k)$ to $z$ and $g$ the loop $\gamma \cdot \psi^k(\gamma^{-1})$, then

$$\gamma \cdot \psi^{kl}(\gamma^{-1}) \simeq g \cdot \psi^k(g) \ldots \psi^{k(l-1)}(g)$$

i.e. both sides have the same area. But the area of the right hand side is $l$ times the area of $g$ (because $\psi$ is symplectic) hence our result.

**Proposition 4.13.** (1) *Assume* $\mathrm{Id} \prec \psi$. *We set* $\mathrm{Per}(N, \psi) = \{$*periodic orbits of* $\psi$ *in* $\mathrm{supp}(\psi)$ *of period* $k$ *with* $1 \leqq k \leqq N\}$. *Then* $\mathrm{card}\,\mathrm{Per}(N, \psi) \geqq C \times N$ *for some* $C = C(\psi)$.

(2) *In general* $\psi$ *has infinitely many periodic points.*

*Proof.* We assume $\psi \neq \mathrm{Id}$ otherwise the result is trivial. Let $c_k = c_+(\psi^k)$ then if $x_k$ is a critical point associated to $c_k$, we have $x_k \neq x_l$ unless $\frac{1}{k}c_k = \frac{1}{l}c_l$. Now let $k(n)$ be a sequence defined inductively as follows:

$$k(n+1)=\inf\left\{k>k(n)\,|\,k>k(n)\times\frac{c_k}{c_{k(n)}}\right\}.$$

Because $c_k \leqq c_\infty \leqq c(\mathrm{supp}(\psi))$, the above sequence is well defined. Moreover because the sequence $\frac{1}{k(n)}c_k(n)$ is strictly decreasing, the points $x_{k(n)}$ are all distinct. Now we wish to estimate $k(n)$. Note that

$$\frac{k(n+1)}{k(n)} \leqq \frac{c_{k(n+1)}}{c_{k(n)}}+\frac{1}{k(n)}.$$

Let $\frac{c_{k(n+1)}}{c_{k(n)}}=1+\varepsilon_n$ we may write

$$k(N)\leqq \prod_{i=1}^{N-1}\left(1+\varepsilon_j+\frac{1}{k(j)}\right)\leqq \prod_{i=1}^{N-1}\left(1+\varepsilon_j+\frac{1}{j}\right)$$
$$\leqq \prod_{i=1}^{N-1}\left(1+\frac{1}{j}\right)\cdot \prod_{i=1}^{N-1}\left(1+\frac{j}{j+1}\varepsilon_j\right).$$

Now $\prod_{i=1}^{\infty}(1+\varepsilon_j)=\frac{c_\infty}{c_1}$ is a convergent product, thus the same is true for

$$\prod_{i=1}^{\infty}\left(1+\frac{j}{j+1}\varepsilon_j\right)$$

since $\varepsilon_j>0$. Let $\gamma$ be its limit, then we have $\gamma \leqq \frac{c_\infty}{c_1}$ and

$$k(N)\leqq \gamma \prod_{i=1}^{N-1}\left(1+\frac{1}{j}\right)=\gamma\cdot N$$

and (1) is proved with $C=\frac{1}{\gamma}\geqq\frac{c_1}{c_\infty}$.

(2) follows from considering an infinite sequence $\psi^{k_n}$ such that $c_+(\psi^{k_n})>0$ (if $k_n$ is free to have any sign, such a sequence always exists). Then because $c_+(\psi^{k_n})$ is bounded, the sequence cannot be made of multiples of a finite number of terms, and the above argument applies. □

We now apply 4.6 to study the continuity of the invariants that we defined.

**Proposition 4.14.** *Let $H_1, H_2$ be two compact supported Hamiltonians. Let $\psi_1, \psi_2$, be the associated time one flows. If $|H_1-H_2|\leqq\varepsilon$, then we have: $|\gamma(\psi_1)-\gamma(\psi_2)|\leqq\varepsilon$. In other words if $C_0(\mathbb{R}^{2n}\times[0,1],\mathbb{R})$ is the set of compact supported $C^2$ Hamiltonians, the map from $C_0(\mathbb{R}^{2n}\times[0,1],\mathbb{R})$ to $\mathbb{R}$, that associates to $H$ the number $\gamma(\psi)$, i.e. the capacity of its time one flow is continuous for the $C^0$ topology.*

*Proof.* Let $H_R$ be a Hamiltonian such that

$$H_R(x)=h_R(|x|),$$

with:

$$h_R(s)=1 \quad \text{for} \quad s\leqq R,$$

$$h_R(s)=0 \quad \text{for} \quad s\geqq 3R,$$

$$|h'(s)|<1 \quad \text{for all } s.$$

We set $H_{R,\varepsilon}=\varepsilon H_R$ for $\varepsilon \leqq 1$, and $\phi_{R,\varepsilon}$ is the time one flow of $H_{R,\varepsilon}$. Note that there is no nonconstant one periodic orbit for $\phi_{R,\varepsilon}$, hence, using Proposition 4.2, we see that $\gamma(\phi_{R,\varepsilon})\leqq 1$. Now, our assumptions imply that $H_2 \leqq H_1+H_{R,\varepsilon}$, and since $H_{R,\varepsilon}$ and $H_1$ have disjoint support, the flow associated to $H_1+H_{R,\varepsilon}$ is $\psi_1 \circ \phi_{R,\varepsilon}$. Using Proposition 4.6, we get the inequality

$$\gamma(\psi_2)\leqq \gamma(\psi_1)+\gamma(\phi_{R,\varepsilon})$$

i.e.

$$\gamma(\psi_2)\leqq \gamma(\psi_1)+\varepsilon .$$

By exchanging $H_1$ and $H_2$, we get the proposition. □

We also have another type of $C^0$ continuity, which answers a question by J. Moser. We would like to thank him for his suggestion.

**Proposition 4.15.** *There is a constant C, independent of R and ε, such that if $\phi$ has support in $B(0,R)$, and $d(x,\varphi(x))\leqq\varepsilon$ for all x, then*

$$\gamma(\varphi)\leqq C\varepsilon R .$$

*More generally, the functions $\gamma, c_+, c_-$ are continuous on $\mathscr{H}^0(\mathbb{R}^{2n})$ for the $C^0$ topology.*

*Remark.* (1) One may explicitely compute $C$, an unchecked computation yields $C=4\pi$.

(2) The above proposition is not symplectically invariant, i.e. if $\varphi$ satisfies the assumptions of the proposition, $\psi\varphi\psi^{-1}$ will not in general. As a result, the above proposition does not imply that $\gamma(\varphi)\leqq C\varepsilon\sqrt{c(\mathrm{supp}(\varphi))}$, or that $\gamma(\varphi)\leqq C\varepsilon R$ whenever $d(x,\varphi(x))\leqq\varepsilon$ and $\mathrm{supp}(\varphi)$ can be embedded in $B(0,R)$.

(3) We shall see that we can have $\gamma(\varphi)=\frac{3}{4}\varepsilon R$.

*Proof of Proposition 4.15.* Let $\psi$ be a compact supported symplectic map. For $z$ a fixed point of $\psi$, we consider $l(z,\psi)$ to be the number $l(z,\infty)$ as defined above in this section, i.e. it is the symplectic area bounded by the loop $c\circ\psi(c^{-1})$, where $c$ is a path from $z$ to $\infty$.

We first need

**Lemma.** *For any $\varepsilon>0$, $R>C\varepsilon$, there exist a symplectic diffeomorphism with compact support such that*

(1) $$\psi \text{ has its support in } B(0,2R),$$

(2) $$d(x,\psi(x))>\varepsilon \quad \text{in} \quad B(0,R),$$

(3) $$0<l(z,\psi)\leqq\frac{C}{2}\varepsilon R \quad \text{for any } z \text{ such that } \psi(z)=z .$$

We postpone the proof of the lemma to conclude the proof of the proposition. According to Proposition 4.4, we may estimate

$$\gamma(\varphi)\leqq\gamma(\psi^{-1})+\gamma(\psi\varphi).$$

According to Proposition 4.2, $c_\pm(\psi\varphi)\leqq 2l(z_\pm,\psi\varphi)$ for some $z_\pm$ satisfying $\psi\varphi(z_\pm)=z_\pm$. We take for $\psi$ the map given by the lemma. Then $\psi\varphi(z)=z$ can only happen if $\varphi(z)=z$ and $\psi(z)=z$. Indeed,

$$d(\varphi(z),\psi(\varphi(z)))=d(z,\varphi(z))\leqq\varepsilon$$

so using (2) $\varphi(z)\notin B(0,R)$ hence $z\notin B(0,R)$ and thus $\varphi(z)=z$.

We may conclude that $l(z, \psi\varphi) = l(z, \psi) \leqq \frac{C_0}{2} \varepsilon R$ so that

$$c_+(\psi\varphi) \leqq \frac{C_0}{2} \varepsilon R, \quad c_-(\psi\varphi) = 0.$$

Using 4.2 once more, we see that $\gamma(\psi^{-1}) \leqq \frac{C_0}{2} \varepsilon R$, thus

$$\gamma(\varphi) \leqq \gamma(\psi^{-1}) + \gamma(\psi\varphi) \leqq \frac{C_0}{2} \varepsilon R + \frac{C_0}{2} \varepsilon R \leqq C_0 \varepsilon R$$

provided $C_0 \varepsilon \leqq R$. On the other hand because the support of $\varphi$ is in $B(0, R)$, we have $\gamma(\varphi) \leqq C(B^{2n}(R)) = \pi R^2$ (follows from Proposition 4.12 and Definition 4.11), so that if $C_0 \varepsilon \geqq R$, $\gamma(\varphi) \leqq \pi R^2 \leqq \pi C_0 \varepsilon R$, thus we may set $C = \pi C_0$. □

*Proof of the lemma.* The idea is to extend the translation by a vector of norm $\varepsilon$ to a symplectic diffeomorphism with support in $B(0, 2R)$. We first notice that if we constructed $\psi_{\varepsilon, R}$ for $\varepsilon \leqq C_0$, $R = 1$, we may set $\psi_{\varepsilon, R}(z) = R\psi_{\varepsilon/R, 1}(z/R)$.

We now set $\psi_{s,1} = \psi_s$.

$\psi_s$ will be the flow of the following Hamiltonian

- $H(x, y) = \delta + ((x, \vec{e}) + 1)$ – where $\vec{e}$ is a unit vector –, if $|x|^2 \leqq 1$, $|y|^2 \leqq \frac{9}{4}$.
- $0 \leqq H(x, y) \leqq 1 + 2\delta$ for all $(x, y)$ in $B(0, 2R)$.
- $H(x, y) = 0$ outside $B(0, 2R)$.

Note that for $(x, y) \in B(0, 1)$, and $s \leqq \frac{1}{2}$, we have $\psi_s(x, y) = (x, y - s\vec{e})$ thus $d(z, \psi_s(z)) \geqq s$. The last step is to prove that $0 < l(z, \psi_s) \leqq Cs$ if $s \leqq \frac{1}{2}$. For this we notice that if $z$ is a fixed point of the flow, i.e. $H'(z) = 0$, then we have that $l(z, \psi_s) = sH(z)$.

Now for $s \leqq s_0$, (where $s_0$ is less than the smallest period of a non trivial closed orbit of the flow, which is strictly positive), $\psi_s(z) = z \Rightarrow H'(z) = 0$ and thus

$$0 \leqq l(z, \psi_s(z)) \leqq s(1 + 2\delta) \leqq \tfrac{3}{2} s$$

for $\delta \leqq \frac{1}{4}$.

Let $C_0 = \sup\left(\frac{1}{s_0}, \frac{3}{2}\right)$ then we have proved

$$\forall \varepsilon, R \quad R \geqq C_0 \varepsilon \text{ we have } d(z, \psi_{\varepsilon/R}(z)) \geqq \varepsilon \text{ and}$$

$$0 \leqq l(z, \psi_{\varepsilon/R}(z)) \leqq \frac{C_0}{2} \varepsilon R \quad \square$$

One may in fact check that $\gamma(\psi_s) \geqq \frac{3}{2} s$, and thus $\gamma(\psi_{\varepsilon, R}) \geqq \frac{3}{2} \varepsilon R$ while $d(z, \psi_{\varepsilon, R}(z)) \leqq 2\varepsilon$ and $\operatorname{supp} \psi_{\varepsilon, R} = B(0, 2R)$. Thus our proposition can not be that much improved.

We leave the following easy generalization to the reader

**Proposition 4.17.** *Given any compact set $W$ in $\mathbb{R}^{2n}$, there is a constant $C(W)$ such that*

$$\forall x \quad d(x, \varphi(x)) \leqq \varepsilon \Rightarrow \gamma(\varphi) \leqq C(W) \varepsilon R$$

*if* $\operatorname{supp}(\varphi) \subset R \cdot W$.

*Remark.* Note that the quantity $C(W)$ is not a symplectic invariant of $W$, it depends also on the metric on $\mathbb{R}^{2n}$. It is however invariant by an isometry of the Kähler structure.

From the above results we get the following

**Corollary 4.18.** *Let $H_n$ be a sequence of Hamiltonians such that their time one flow $\psi_n$ converges in the $C^0$ topology to a symplectic map $\psi$. Then if $H_n$ converges in the $C^0$ topology to a Hamiltonian $H$ with time one flow $\phi$, then we have $\gamma(\psi)=\gamma(\phi)$ (and also $c_+(\psi)=c_+(\phi)$, $c_-(\psi)=c_-(\phi)$).*

**Note added in proof.** Corollary 4.18 may be improved to conclude that $\varphi=\psi$. Indeed, $\varphi^{-1}\psi_n$ is the time one flow of $H_n(t,\varphi_t(z))-H(t,\varphi_t(z))$. Since this goes to zero in the $C^0$ topology, $\varphi^{-1}\psi_n$ goes to the identity, hence $\varphi=\psi$.

Using the fact that $\gamma(\phi)=0$ only for $\phi=\mathrm{Id}$ this implies as a special case the following result by Hofer:

**Corollary 4.19** (cf. [H]). *Let $H_n$ be a sequence of compact supported Hamiltonians such that $H_n \xrightarrow[C^0]{} 0$, and their time one maps $\psi_n$ converge $C^0$ to some map $\psi$. Then $\psi=\mathrm{Id}$.*

## 5 On the effect of symplectic reduction and the camel problem

Consider two compact manifolds $V, W$, and let $L$ be a Lagrange submanifold of $T^*(V\times W)$. For almost every $w$ in $W$, $L$ is transverse to $T^*V\times T^*_wW$, so that we may define the reduction $L_w$ of $L\cap T^*V\times T^*_wW$.

Now if $L$ has the g.f.q.i. $S: E\to\mathbb{R}$ defined on the vector bundle $E\xrightarrow{\pi} V\times W$, then $L_w$ has the g.f.q.i. $S_w=S_{|\pi^{-1}(V\times\{w\})}: E_w\to\mathbb{R}$.

Note that $L_w$ is not necessarily isotopic to the zero section, even if $L$ is. In this case we cannot speak about $c(\alpha, L_w)$, since the number in Definition 2.1 might depend on the choice of a generating function for $L_w$, however we may define $c(\alpha, S_w)$.

We may now state

**Proposition 5.1.** *For $\alpha$ in $H^*(V)$, we have, if $S$ is a g.f.q.i. for $L$,*

$$\begin{aligned} c(\alpha\otimes 1, L) &\leqq \inf_w c(\alpha, S_w) \\ &\leqq \sup_w c(\alpha, S_w) \leqq c(\alpha\otimes\mu_W, L). \end{aligned}$$

*Proof.* Consider the sequences

$$\begin{array}{ccccc} H^*(V\times W) & \xrightarrow{T} & H^*(E^\infty, E^{-\infty}) & \longrightarrow & H^*(E^\lambda, E^{-\infty}) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(V) & \xrightarrow{T} & H^*(E_w^\infty, E_w^{-\infty}) & \longrightarrow & H^*(E_w^\lambda, E_w^{-\infty}) \end{array}$$

where the map $H^*(V\times W)\to H^*(V)$ is induced by the injection $V\to V\times\{w\}\subset V\times W$, and coincides with the composition of the projection on $H^*(V)\otimes H^0(W)$ and the obvious identification $H^*(V)\otimes H^0(W)\to H^*(V)$.

Now if the image of $T(\alpha)$ is nonzero in $H^*(E_w^\lambda, E_w^{-\infty})$ that is for $\lambda\geqq c(\alpha, L_w)$, we must have that the image of $T(\alpha\otimes 1)$ in $H^*(E^\lambda, E^{-\infty})$ is nonzero, that is $\lambda\geqq c(\alpha\otimes 1, L)$. Thus we proved that if $\lambda\geqq c(\alpha, S_w)$ then $\lambda\geqq c(\alpha\otimes 1, L)$ that is $c(\alpha, S_w)\geqq c(\alpha\otimes 1, L)$. Since this holds for every $w\in W$ we get the first inequality. The second inequality is trivial, and the third one is obtained from the first one by changing $L$ to $\bar{L}$.

We now consider an open set $U$ in $\mathbb{R}^{2(n+m)}$, and the family of its reductions,

$$U_w=(U\cap\mathbb{R}^{2n}\times\mathbb{R}^m\times\{0\})/\{0\}\times\mathbb{R}^m.$$

We wish to estimate $c(U)$ using the $U_w$.

First of all we slightly change our familiar setting. If $\psi$ is in $\mathscr{H}^0(\mathbb{R}^{2(n+m)})$, we consider again $\Gamma_\psi$ in $T^*\Delta_{\mathbb{R}^{2n}\times\mathbb{R}^{2m}}$. Now instead of compactifying $\mathbb{R}^{2n}\times\mathbb{R}^{2m}$ to $S^{2(n+m)}$, we may compactify it to $S^{2n}\times S^{2m}$, and $\Gamma_\psi$ becomes a Lagrange submanifold of $T^*(S^{2n}\times S^{2m})$. We may thus consider four numbers, namely $c(1\otimes 1,\Gamma_\psi)$, $c(1\otimes\mu_{S^{2m}},\Gamma_\psi)$, $c(\mu_{S^{2n}\otimes 1},\Gamma_\psi)$, and $c(\mu_{S^{2n}}\otimes\mu_{S^{2n}},\Gamma_\psi)$. Accordingly, as in Definition 4.11, we may define $c(\alpha,U)$, $\gamma(\alpha,U)$ for $\alpha$ one of the above four cohomology classes. □

We shall first need to slightly extend several notions that have been introduced in the previous sections.

First of all, we may generalize the capacities defined in Sect. 4 to the case of compact supported Hamiltonian isotopies of $\mathbb{R}^{2n}\times T^*T^k$ ($T^k$ is the $k$ dimensional torus).

This is easy if we notice that such isotopy, $\psi_t$, may be lifted to an isotopy $\tilde{\psi}_t$ of $\mathbb{R}^{2n}\times\mathbb{R}^{2m}$ such that $\tilde{\psi}_t(x+v)=\tilde{\psi}_t(x)+v$ for $v\in\{0\}\times\mathbb{Z}^k$.

Thus $\Gamma_{\tilde{\psi}_1}$ is such that if $(x,y)\in\Gamma_{\tilde{\psi}}$, then $(x+v,y+v)\in\Gamma_{\tilde{\psi}}$, this means that

$$\Gamma_{\tilde{\psi}}\subset\overline{\mathbb{R}^{2n}}\times\overline{\mathbb{R}^{2k}}\times\mathbb{R}^{2n}\times\mathbb{R}^{2k}=T^*(\Delta_{\mathbb{R}^{2n}\times\mathbb{R}^{2k}})$$

descends to the quotient $T^*(\Delta_{\mathbb{R}^{2n+k}}\times T^k)$.

Let $\tilde{\Gamma}_\psi$ be the Lagrange submanifold thus obtained, it is easy to see that $\tilde{\Gamma}_\psi$ coincides with the zero section outside a compact set (because $\psi_t$ is compact supported).

We may again compactify $T^*(\mathbb{R}^{2n+k}\times T^k)$ into $T^*(S^{2n+k}\times T^k)$, and $\tilde{\Gamma}_\psi$ may be compactified into $\Gamma_\psi$ in $T^*(S^{2n+k}\times T^k)$. Now, if $\alpha\otimes\beta$ is a cohomology class in $H^*(S^{2n+k})\otimes H^*(T^k)$, we may define $c(\alpha\otimes\beta,\Gamma_\psi)$.

We may now prove

**Proposition 5.2.** $c(\mu_{S^{2n}}\otimes 1,U)\leqq\inf_w\gamma(U_w)$.

*Proof.* It is clear from 5.1 that the following is true:

$$c(\mu_{S^{2n}}\otimes 1,\psi)\leqq\inf_w c(\mu,S_w)$$

where $S$ is a g.f.q.i. for $\Gamma_\psi$ and $\psi$ is in $\mathscr{H}^0(U)$.

Now let $\psi_t$ be a symplectic isotopy supported in $U$ with $\psi_0=\mathrm{id}$, $S^t$ a g.f.q.i. for $\Gamma_{\psi_t}$, and $S_w^t$ its restriction to $\pi^{-1}(S^{2n}\times\{w\})$. We are going to show that $c(\mu,S_w^1)\leqq\gamma(U_w)$.

Let $\varphi$ be an element in $\mathscr{H}^0(\mathbb{R}^{2n})$ such that $\varphi(U_w)\cap U_w=\emptyset$. We denote by $\varphi S_w^t$ the generating function of $\varphi L_w^t$ obtained from $S_w^t$. Now the proof of Proposition 3.5 applies, and we may state with $\tilde{\varphi}=\varphi\times\mathrm{id}$

$$c(\alpha,\tilde{\varphi}S_w)=c(\alpha,S_w-\tilde{\varphi}^{-1}(O_{S^{2n}}))$$

where $\tilde{\varphi}^{-1}(O_{S^{2n}})$ denotes any g.f.q.i. of this manifold.

Because $c(\alpha,\tilde{\varphi}S_w^t)$ is one of the numbers $l(x,y)$ associated to $\tilde{\varphi}L_w^t$, and $L_w^t$ coincides with $\Delta_w$ outside $U_w\times U_w$, the proof of Corollary 4.5 carries over, thus concluding our proof. □

The "camel problem" can be stated as follows: there is no symplectic isotopy, $\psi_t$, with support in $(\mathbb{R}^{2n}-\mathbb{R}^{2n-1})\cup B^{2n}(\varepsilon)$, such that $\psi_0=\mathrm{id}$ and $\psi_1$ sends a ball of radius $r>\varepsilon$ contained in one component of $\mathbb{R}^{2n}-\mathbb{R}^{2n-1}$ in to the other component.

In other terms a ball of radius $r > \varepsilon$ cannot "go through a hole of radius $\varepsilon$" (see also [Ar, V3] for historical details). In fact this may be generalized as follows: let $V \subset \mathbb{R}^{2n-1}$ be such that its reduction has capacity $\gamma(V/\mathbb{R})$. Then $U$ cannot go through $\mathbb{R}^{2n-1} - V$ unless $c(U) \leqq \gamma(V/\mathbb{R})$.

To keep our proof simple, we shall only consider the case of balls, but with a little more work, we could easily extend our proof to this more general case.

An alternative (unpublished) proof is due to Gromov, and Eliashberg using the technique developed in [Gr1]. I wish to thank them for enlightening discussions.

We now go back to the camel problem. Let $\psi_t$ be a compact supported Hamiltonian isotopy such that $\psi_1$ sends a ball $B_r$ of radius $r$ from one side to the other of the hyperplane $\mathbb{R}^{2n-1} \times \{0\}$. We then have

**Proposition 5.3.** *Let $V$ be $\bigcup_{t \in [0,1]} \psi_t(B_r) \cap \mathbb{R}^{2n-1} \times \{0\}$, and $V/\mathbb{R}$ be the quotient of $V$ by the characteristic foliation in $\mathbb{R}^{2n-1}$.*

*Then $V/\mathbb{R}$ is a subset of $\mathbb{R}^{2n-2}$ such that $\gamma(V/\mathbb{R}) \geqq \pi r^2$.*

Before we prove the above proposition, we shall make some simplifying assumptions.

We take on $\mathbb{R}^{2n}$, the coordinates $(q, p, q_n, p_n) \in \mathbb{R}^{2n-2} \times \mathbb{R}^2$, the hyperplane $\mathbb{R}^{2n-1} \times \{0\}$ is then given by $\{p_n = 0\}$.

Without modifying $V = \bigcup_t \psi_t(B_r) \cap \mathbb{R}^{2n-1} \times \{0\}$, we may assume that

(1) For some $c \gg r$ we have

$$\psi_t(q, p, q_n, p_n + c) = \psi_t(q, p, q_n, p_n) + c\frac{\partial}{\partial p_n},$$

$$\psi_1(q, p, q_n, p_n) = (q, p, q_n, p_n + c),$$

(2) $\psi_t$ is defined for all $t$'s and $\psi_{t+1} = \psi_1 \circ \psi_t$.

This follows easily from the fact that all symplectic embeddings of $B_r$ into the half space $\mathbb{R}^{2n-1} \times \mathbb{R}_+$ are symplectically isotopic, and from the extension of isotopies.

Now let $H(t, q, p, q_n, p_n)$ be the Hamiltonian generating $\psi_t$. We shall define a symplectic map $\bar{\Psi}: \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ by

$$\bar{\Psi}(q, p, q_n, p_n, t, h) = (\psi_t(q, p, q_n, p_n), t, h + H(t, q, p, q_n, p_n))$$

We infer from (1) and (2) that

$$\bar{\Psi}(q, p, q_n, p_n + kc, t + l, h) = \bar{\Psi}(q, p, q_n, p_n) + kc\frac{\partial}{\partial p_n} + l\frac{\partial}{\partial t}.$$

This means that on $\mathbb{R}^{2n-2} \times T^*T^2 \simeq \mathbb{R}^{2n+2} \Big/ \left(\mathbb{Z}c\frac{\partial}{\partial p_n} + \mathbb{Z}\frac{\partial}{\partial t}\right)$, $\bar{\Psi}$ descends to a map $\Psi = \Psi_1$.

If we denote by $\Psi_0$ the map obtained above by taking

$$\psi_t(q, p, q_n, p_n) = (q, p, q_n, p_n + tc),$$

it is easy to see that $\Psi_1$ is symplectically isotopic to $\Psi_0$. □

Now let $U(r)$ be $\Psi_1(B^{2n}(r) \times T^*S^1)$, we have

**Lemma 5.4.** $c(\mu \otimes d(p_n + ct) \otimes 1, U(r)) \geqq \pi r^2$.

*Proof.* For simplicity, we assume $c=1$. Clearly

$$\begin{aligned}
&c(\mu\otimes d(p_n+t)\otimes 1, \Psi_1(B^{2n}(r)\times T^*S^1))\\
&=c(\mu\otimes d(p_n+t)\otimes 1, \Psi_0(B^{2n}(r)\times T^*S^1))\\
&=c(\Psi_0^*(\mu\otimes d(p_n+t)\otimes 1), B^{2n}(r)\times T^*S^1)\\
&=c(\mu\otimes dp_n\otimes 1, B^{2n}(r)\times T^*S^1)\\
&=c(\mu\otimes dp_n, B^{2n}(r))=\pi r^2\,.
\end{aligned}$$

**Lemma 5.5.**

$$c(\mu\otimes d(p_n+t)\otimes 1, U(r))\leqq\gamma\left(\mu, \bigcup_{t\in[0,1]}\psi_t(B^{2n}(r))\cap\mathbb{R}^{2n-1}\times\{0\}/\mathbb{R}\right)=\gamma(\mu, V/\mathbb{R})\,.$$

*Proof.* If we remark that the torus $T^2=\{(p_n,t)\in S^1\times S^1\}$ can also be written as the product of the circles $\{p_n+t=0\}$ and $\{p_n=0\}$, we see that according to Proposition 5.2, we may infer that

$$\begin{aligned}
c(\mu\otimes d(p_n+t)\otimes 1, U(r))&\leqq\inf_s\gamma(\mu\otimes d(p_n+t), U(r)\cap\{p_n=s\}/\mathbb{R})\\
&\leqq\gamma(\mu\otimes d(p_n+t), U(r)\cap\{p_n=0\}/\mathbb{R})\,.
\end{aligned}$$

Now

$$U(r)\cap\{p_n=0\}/\mathbb{R}\subset\left(\bigcup_{t\in[0,1]}\psi_t(B^{2n}(r))\cap\mathbb{R}^{2n-1}\times\{0\}/\mathbb{R}\right)\times T^*S^1=V/\mathbb{R}\times T^*S^1\,.$$

Since $\gamma(\mu\otimes d(p_n+t), V/\mathbb{R}\times T^*S^1)=\gamma(\mu, V/\mathbb{R})$ according to 5.2, this concludes our proof. □

*Remark.* The same proof yields the following result. Let

$$H^k(\varepsilon)=(\mathbb{R}^{2n}-\mathbb{R}^n\times\mathbb{R}^k)\cup B^{2n}(\varepsilon)$$

be the complement of a punctured coisotropic subspace in $\mathbb{R}^{2n}$, and $\mathrm{SEmb}(B^{2n}(r), H^k(\varepsilon))$ be the space of symplectic embeddings of the ball in $H^k(\varepsilon)$, $\partial\mathrm{SEmb}(B^{2n}(r), H^k(\varepsilon))$ be the subspace of those embeddings with their image outside a large ball of $\mathbb{R}^{2n}$. Then one may prove that

$$\pi_{k-1}(\partial\,\mathrm{SEmb}(B^{2n}(r), H^k(\varepsilon))\to\pi_{k-1}(\mathrm{SEmb}(B^{2n}(r), H^k(\varepsilon)))$$

is zero if and only if $r\leqq\varepsilon$.

## 6 On some properties of simple manifolds

We consider the hypersurfaces introduced by Eliashberg in [El], which satisfy the following two properties

(i) $\Sigma$ coincides with $\Sigma_0=\{(q,q_n,p,p_n)\,|\,p_n=0\}$ outside $U\times[0,1]\times\mathbb{R}$ ($U$ is a compact subset of $\mathbb{R}^{2n-2}$),

(ii) a characteristic of $\Sigma$ which intersects $U\times\{0\}\times\{0\}$, also intersects $U\times\{1\}\times\{0\}$.

We shall usually summarize (ii) by saying that all characteristics of $\Sigma$ "go from one side to the other", and call a hypersurface satisfying the above conditions, simple.

To a simple hypersurface $\Sigma$, one may associate

– a symplectic diffeomorphism of $\mathbb{R}^{2n-2}$, $\phi_\Sigma$, which coincides with the identity outside $U$, defined by the property: $y=\phi_\Sigma(x)\Leftrightarrow(x,0,0)$ and $(y,1,0)$ are on the same characteristic of $\Sigma$.

– a Lagrange submanifold of $\overline{\mathbb{R}^{2n-2}} \times \mathbb{R}^{2n}$,

$$\mathscr{L}_\Sigma = \{(x, y) \in \overline{\mathbb{R}^{2n-2}} \times \mathbb{R}^{2n} \mid y \in \Sigma, \text{ and is on the characteristic of } \Sigma \text{ which goes through } (x, 0, 0)\}.$$

Note that the reduction of $\mathscr{L}_\Sigma$ by $\{q_n = 1\}$ is the graph of $\varphi_\Sigma$. Note also that if $H(t, x)$ is a time dependent compact supported Hamiltonian on $\mathbb{R}^{2n-2}$, with time one flow $\varphi$, then for

$$\Sigma = \{(x, t, h) \in \mathbb{R}^{2n} \mid h = H(t, x)\}$$

we have $\varphi_\Sigma = \varphi$.

[Remark: We had to assume $H(t, x) = 0$ for $t \notin [0, 1]$, but we can always do so by replacing $H(t, x)$ by $H(t, x) - a(t)$, since $H(t, x)$ does not depend on $x$ for $t \notin [0, 1]$.]

We now, as in Sect. 4, replace $\overline{\mathbb{R}^{2n-2}} \times \mathbb{R}^{2n}$ by $T^*\mathbb{R}^{2n-1}$ using the map

$$(q, p, Q, Q_n, P, P_n) \to \left(\frac{q+Q}{2}, \frac{p+P}{2}, Q_n, P - p, q - Q, P_n\right).$$

Note that for $\Sigma_0 = \mathbb{R}^{2n-1} \times \{0\}$, $\mathscr{L}_0 = \mathscr{L}_{\Sigma_0}$ goes to the zero section. We now compactify $\mathscr{L}_\Sigma$ and the zero section $\mathscr{L}_0$ to $S^{2n-2} \times S^1$, the $S^1$ direction being given by $Q_n$.

In order to do this we need that $\mathscr{L}_\Sigma = \mathscr{L}_{\Sigma_0}$ outside a compact set, and this happens if and only if $\varphi_\Sigma = \mathrm{id}$. This is slightly unpleasant, since we want to relate $\varphi_\Sigma$ to $\Sigma$, so we operate as follows: change $\Sigma$ on $U \times [1, 2] \times \{0\}$ (where $\mathscr{L}_\Sigma = \mathscr{L}_{\Sigma_0}$) to the graph of a Hamiltonian $K(t, x)$ such that its time one map is $\varphi_\Sigma^{-1}$. We thus get a simple hypersurface $\Sigma'$. We may now compactify simultaneously $\mathscr{L}_{\Sigma_0}$ and $\mathscr{L}_{\Sigma'}$, and get a Lagrange submanifold $\overline{\mathscr{L}_{\Sigma'}}$ of $T^*\overline{L_{\Sigma_0}} = T^*(S^{2n-2} \times S^1)$.

**Definition 6.1.** For

$$\alpha \otimes \beta \in H^*(S^{2n-2} \times S^1) = H^*(S^{2n-2}) \otimes H^*(S^1),$$

we set $c(\alpha \otimes \beta, \Sigma')$ for $c(\alpha \otimes \beta, L_{\Sigma'})$.

However if $\Sigma'$ is obtained from $\Sigma$ as explained above, we keep the notation $\varphi_\Sigma$ for the symplectomorphism (usually not the identity) defined before.

We may prove

**Proposition 6.2.**

$$c(1 \otimes 1, \Sigma') \leqq c_-(\varphi_\Sigma) \leqq c(1 \otimes \mu, \Sigma'),$$

$$c(\mu \otimes 1, \Sigma') \leqq c_+(\varphi_\Sigma) \leqq c(\mu \otimes \mu, \Sigma').$$

*Proof.* Follows immediately from Proposition 5.1 and the fact that $\overline{\Gamma}_{\varphi_\Sigma}$ is the reduction of $\mathscr{L}_{\Sigma'}$ by $\{Q_n = 1\}$. □

Remember that Corollary 4.7 told us that if $\varphi$ is the flow of a nonnegative Hamiltonian, then $c_-(\varphi) = 0$. We wish to generalize this statement as follows.

**Proposition 6.3.** *Assume $\Sigma$ to be contained in $\mathbb{R}^{2n-1} \times \mathbb{R}_+$. Then $c_-(\varphi_\Sigma) = 0$, i.e. $\varphi_\Sigma \succ \mathrm{id}$.*

*Proof.* Let $(x, y, Q_n)$ be coordinates on $\overline{\mathscr{L}_{\Sigma_0}}$, $(x, y, Q_n, \xi, \eta, P_n)$ be coordinates in $T^*\overline{\mathscr{L}_{\Sigma_0}}$. Let $S(x, y, Q_n, \zeta)$ be a g.f.q.i. for $\overline{\mathscr{L}_{\Sigma'}}$, then $S(x, y, 1, \zeta)$ is a g.f.q.i. for $\overline{\Gamma_{\varphi_\Sigma}}$, and $S(x, y, Q_n, \xi)$ is a quadratic form in $\xi$ for $Q_n = 0$. For each $\theta \in S^1$, denote by $S_\theta(x, y, \xi)$ the restriction of $S$ to $\{Q_n = \theta\}$. Then $c(1, S_\theta)$ is zero for $\theta = 0$, is equal to $c_-(\varphi_\Sigma)$ for $\theta = 1$ and since $\frac{\partial S_\theta}{\partial \theta} = \frac{\partial S}{\partial Q_n} = P_n \geqq 0$ for $\theta \in [0, 1]$ if $\frac{\partial S_\theta}{\partial \xi}(x, y, S) = 0$, we get as in the proof of Proposition 4.6 that $c(1, S_\theta)$ is a nondecreasing function on $[-1, 0]$. But since $c(1, S_0) = 0$ and $c_-(\varphi_\Sigma) \leqq 0$, we clearly get $c_-(\varphi_\Sigma) = 0$. □

*Acknowledgements.* It is a pleasure to thank Yasha Eliashberg and Dusa McDuff for many useful conversations. I also warmly thank, François Laudenbach for chasing an uncountable number of mistakes, and suggesting many improvements, and Marc Chaperon for simplifying the proof of Proposition 4.6. Most of this paper was written while I was visiting the Forschungsinstitut für Mathematik of the E.T.H. in Zürich. I would like to thank Prof. Jürgen Moser for his kind invitation. I am also very grateful to Edi Zehnder with whom I had many useful discussions and who patiently listened to a preliminary version of some of the results of this paper.

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